

# PURSUING THE DOUBLE AFFINE GRASSMANNIAN I: TRANSVERSAL SLICES VIA INSTANTONS ON $A_k$ -SINGULARITIES

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*To the memory of Izrail Moiseevich Gelfand*

**ABSTRACT.** This paper is the first in a series that describe a conjectural analog of the geometric Satake isomorphism for an affine Kac-Moody group (in this paper for simplicity we consider only untwisted and simply connected case). The usual geometric Satake isomorphism for a reductive group  $G$  identifies the tensor category  $\text{Rep}(G^\vee)$  of finite-dimensional representations of the Langlands dual group  $G^\vee$  with the tensor category  $\text{Perv}_{G(\mathcal{O})}(\text{Gr}_G)$  of  $G(\mathcal{O})$ -equivariant perverse sheaves on the affine Grassmannian  $\text{Gr}_G = G(\mathcal{K})/G(\mathcal{O})$  of  $G$  (here  $\mathcal{K} = \mathbb{C}((t))$  and  $\mathcal{O} = \mathbb{C}[[t]]$ ). As a byproduct one gets a description of the irreducible  $G(\mathcal{O})$ -equivariant intersection cohomology sheaves of the closures of  $G(\mathcal{O})$ -orbits in  $\text{Gr}_G$  in terms of  $q$ -analogs of the weight multiplicity for finite dimensional representations of  $G^\vee$ .

The purpose of this paper is to try to generalize the above results to the case when  $G$  is replaced by the corresponding affine Kac-Moody group  $G_{\text{aff}}$  (we shall refer to the (not yet constructed) affine Grassmannian of  $G_{\text{aff}}$  as the *double affine Grassmannian*). More precisely, in this paper we construct certain varieties that should be thought of as transversal slices to various  $G_{\text{aff}}(\mathcal{O})$ -orbits inside the closure of another  $G_{\text{aff}}(\mathcal{O})$ -orbit in  $\text{Gr}_{G_{\text{aff}}}$ . We present a conjecture that computes the IC sheaf of these varieties in terms of the corresponding  $q$ -analog of the weight multiplicity for the *Langlands dual affine group*  $G_{\text{aff}}^\vee$  and we check this conjecture in a number of cases.

Some further constructions (such as convolution of the corresponding perverse sheaves, analog of the Beilinson-Drinfeld Grassmannian etc.) will be addressed in another publication.

## 1. INTRODUCTION

**1.1. Langlands duality and the Satake isomorphism.** Let  $F$  be a global field and let  $\mathbb{A}_F$  denote its ring of adeles. Let  $G$  be split reductive group over  $F$ . The classical *Langlands duality* predicts that irreducible automorphic representations of  $G(\mathbb{A}_F)$  are closely related to the homomorphisms from the absolute Galois group  $\text{Gal}_F$  of  $F$  to the *Langlands dual group*  $G^\vee$ . Similarly, if  $G$  is a split reductive group over a local-nonarchimedean field  $\mathcal{K}$ , Langlands duality predicts a relation between irreducible representations of  $G(\mathcal{K})$  and homomorphisms from  $\text{Gal}_{\mathcal{K}}$  to  $G^\vee$ .

The starting point for Langlands duality is the following *Satake isomorphism*. Let  $\mathcal{O} \subset \mathcal{K}$  denote the ring of integers of  $\mathcal{K}$ . Then the group  $G(\mathcal{K})$  is a locally compact topological group and  $G(\mathcal{O})$  is its maximal compact subgroup. One may study the *spherical Hecke algebra*  $\mathcal{H}$  of  $G(\mathcal{O})$ -biinvariant compactly supported  $\mathbb{C}$ -valued measures on  $G(\mathcal{K})$ . The Satake isomorphism is a canonical isomorphism between  $\mathcal{H}$  and the complexified Grothendieck ring  $K_0(\text{Rep}(G^\vee))$  of finite-dimensional representations of  $G^\vee$ .

**1.2. The geometric version.** The classical Langlands duality has its geometric counterpart, usually referred to as the *geometric Langlands duality*. It is based on the following geometric version of the Satake isomorphism. Let now  $\mathcal{K} = \mathbb{C}((s))$  and let  $\mathcal{O} = \mathbb{C}[[s]]$ ; here  $s$  is a formal variable. Let  $\mathrm{Gr}_G = G(\mathcal{K})/G(\mathcal{O})$ . Then the geometric analog of the algebra  $\mathcal{H}$  considered above is the category  $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$  of  $G(\mathcal{O})$ -equivariant perverse sheaves on  $\mathrm{Gr}_G$  (cf. [13], [21] or [22] for the precise definitions). According to *loc. cit.* the category  $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$  possesses canonical tensor structure and the geometric Satake isomorphism asserts that this category is equivalent to  $\mathrm{Rep}(G^\vee)$  as a tensor category. The corresponding fiber functor from  $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$  to vector spaces sends every perverse sheaf  $\mathcal{S} \in \mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$  to its cohomology.

More precisely, one can show (we review it in Section 2) that  $G(\mathcal{O})$ -orbits on  $\mathrm{Gr}_G$  are finite-dimensional and they are indexed by the set  $\Lambda^+$  of dominant weights of  $G^\vee$ . For every  $\lambda \in \Lambda^+$  we denote by  $\mathrm{Gr}_G^\lambda$  the corresponding orbit and by  $\overline{\mathrm{Gr}}_G^\lambda$  its closure in  $\mathrm{Gr}_G$ . Then  $\mathrm{Gr}_G^\lambda$  is a non-singular quasi-projective algebraic variety over  $\mathbb{C}$  and  $\overline{\mathrm{Gr}}_G^\lambda$  is a (usually singular) projective variety. One has

$$\overline{\mathrm{Gr}}_G^\lambda = \bigcup_{\mu \leq \lambda} \mathrm{Gr}_G^\mu.$$

One of the main properties of the geometric Satake isomorphism is that it sends the irreducible  $G^\vee$ -module  $L(\lambda)$  to the intersection cohomology complex  $\mathrm{IC}(\overline{\mathrm{Gr}}_G^\lambda)$ . In particular, the module  $L(\lambda)$  itself gets realized as the intersection cohomology of the variety  $\overline{\mathrm{Gr}}_G^\lambda$ .

As a byproduct of the geometric Satake isomorphism one can compute  $\mathrm{IC}(\overline{\mathrm{Gr}}_G^\lambda)$  in terms of the module  $L(\lambda)$ . Namely, it follows from [21], [8] and [13] that the stalk of  $\mathrm{IC}(\overline{\mathrm{Gr}}_G^\lambda)$  at a point of  $\mathrm{Gr}_G^\mu$  as a graded vector space is essentially equal to the associated graded  $\mathrm{gr}^F L(\lambda)_\mu$  of the  $\mu$ -weight space  $L(\lambda)_\mu$  in  $L(\lambda)$  with respect to certain filtration (called sometimes *the Brylinski-Kostant filtration*). We refer the reader to Section 2.4 for more details.

In fact, in [16] the authors construct certain canonical transversal slice  $\overline{\mathcal{W}}_{G,\mu}^\lambda$  to  $\mathrm{Gr}_G^\mu$  inside  $\overline{\mathrm{Gr}}_G^\lambda$ . This is a conical affine algebraic variety (i.e. it is endowed with an action of the multiplicative group  $\mathbb{G}_m$  which contracts it to one point). The above result about the stalks of  $\mathrm{IC}(\overline{\mathrm{Gr}}_G^\lambda)$  then gets translated into saying that the stalk of the IC-sheaf of  $\overline{\mathcal{W}}_{G,\mu}^\lambda$  at the unique  $\mathbb{G}_m$ -fixed point is essentially isomorphic to  $\mathrm{gr}^F L(\lambda)_\mu$ . Note that  $\overline{\mathcal{W}}_{G,\mu}^\lambda$  is contracted to the above point by the  $\mathbb{G}_m$ -action, it follows that the stalk of  $\mathrm{IC}(\overline{\mathcal{W}}_{G,\mu}^\lambda)$  is equal to the global intersection cohomology  $\mathrm{IH}^*(\overline{\mathcal{W}}_{G,\mu}^\lambda)$ .

**1.3. The affine case.** This paper should be considered as a part of an attempt to generalize Langlands duality to the case of affine Kac-Moody groups. More precisely, in this paper we generalize some of the facts related to the geometric Satake isomorphism to the affine case. Let us mention that simultaneously in [7] an attempt is made also to generalize the more classical story (such as the usual Satake isomorphism) to the affine case.

Let us now assume that  $G$  is semi-simple and simply connected and let  $G_{\mathrm{aff}}$  denote the corresponding (untwisted) affine Kac-Moody group (cf. Section 3); let  $\mathfrak{g}_{\mathrm{aff}}$  be the Lie algebra of  $G_{\mathrm{aff}}$ . In Section 3 we explain that to  $G_{\mathrm{aff}}$  one can attach the corresponding dual affine

Kac-Moody group  $G_{\text{aff}}^\vee$ . The Lie algebra  $\mathfrak{g}_{\text{aff}}^\vee$  of  $G_{\text{aff}}^\vee$  is an affine Kac-Moody algebra whose Dynkin diagram is dual to that of  $\mathfrak{g}_{\text{aff}}$ . If  $\mathfrak{g}$  is simply laced, the algebra  $\mathfrak{g}_{\text{aff}}^\vee$  is isomorphic to  $\mathfrak{g}_{\text{aff}}$ . In general, however, this might be a twisted affine Lie algebra (in particular, in general  $\mathfrak{g}_{\text{aff}}^\vee \not\cong (\mathfrak{g}^\vee)_{\text{aff}}$ ).

The idea of this paper came to us from a conversation with I. Frenkel who suggested that integrable representations of  $G_{\text{aff}}$  of level  $k$  should be realized geometrically in terms of some moduli spaces related to  $G$ -bundles on  $\mathbb{A}^2/\Gamma_k$ , where  $\Gamma_k$  is the group of roots of unity of order  $k$  acting on  $\mathbb{A}^2$  by  $\zeta(x, y) = (\zeta x, \zeta^{-1}y)$ .<sup>1</sup> In this paper we give some kind of rigorous meaning to this suggestion with a small modification:  $G_{\text{aff}}$  has to be replaced by  $G_{\text{aff}}^\vee$ .<sup>2</sup> Roughly speaking, this paper constitutes an attempt to make precise the following principle. Let  $\text{Bun}_G(\mathbb{A}^2)$  denote the moduli space of principal  $G$ -bundles on  $\mathbb{P}^2$  trivialized at the “infinite” line  $\mathbb{P}_\infty^1 \subset \mathbb{P}^2$ . This is an algebraic variety which has connected components parametrized by non-negative integers, corresponding to different values of the second Chern class of the corresponding bundles. Similarly, one can define  $\text{Bun}_G(\mathbb{A}^2/\Gamma_k)$  (cf. Section 4.4 for the details). Very vaguely, the main idea of this paper can be formulated in the following way:

### The basic principle:

1) The integrable representations of  $G_{\text{aff}}^\vee$  (not of  $G_{\text{aff}}$ !) of level  $k$  have to do with the geometry (e.g. intersection cohomology) of some varieties closely related to  $\text{Bun}_G(\mathbb{A}^2/\Gamma_k)$ .

2) This relation should be thought of as similar to the relation between finite-dimensional representations of  $G^\vee$  and the geometry of the affine Grassmannian  $\text{Gr}_G$ .

We believe that 1) above has many different aspects. In this paper we concentrate on just one such aspect; namely, in Section 4 we explain how one can construct an analog of the varieties  $\overline{\mathcal{W}}_\mu^\lambda$  in the affine case (using the variety  $\text{Bun}_G(\mathbb{A}^2/\Gamma_k)$  as well as the corresponding *Uhlenbeck compactification* of the moduli space of  $G$ -bundles - cf. [4] and Section 4). We conjecture that the stalks of IC-sheaves of these varieties are governed by the affine version of  $\text{gr}^F L(\lambda)_\mu$  (cf. Conjecture 4.14).

Let us note that in the case  $G = \text{SL}(n)$  a slightly different approach to 1) is discussed in [20]. Also, I. Grojnowski has informed us that he made some attempts to construct  $\text{Gr}_{G_{\text{aff}}}$  (apparently, using completely different methods).

We do not know how to prove our main conjecture in general; in the last three sections of the paper we check our conjecture in several special cases. In particular, in Section 5 we prove all our conjectures in the limit  $k \rightarrow \infty$  (cf. Section 5 for the exact formulation). In Section 6 we prove a slightly weaker version of our main Conjecture 4.14 in the case  $k = 1$ ; the proof is based on the results of [4]. Also in Section 7 we prove again a slightly weaker form of our conjecture for  $G = \text{SL}(N)$ . Let us mention the main ingredient of that proof. Let  $\mathfrak{g}$  be a simply laced simple finite-dimensional Lie algebra. Then (by McKay correspondence) to  $\mathfrak{g}$  one can associate a finite subgroup  $\Gamma$  of  $\text{SL}(2, \mathbb{C})$ . Recall that H. Nakajima (cf. e.g. [24]) gave a geometric construction of integrable  $\mathfrak{g}_{\text{aff}}$ -modules of level  $N$  using certain moduli

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<sup>1</sup>I. Frenkel’s suggestion is discussed also in [20] whose relation to this paper is not clear to us at the moment.

<sup>2</sup>In fact, I. Frenkel made this suggestion only when  $G$  is simply laced, in which case there is almost no difference between  $G_{\text{aff}}$  and  $G_{\text{aff}}^\vee$ .

spaces which, roughly speaking, have to do with vector bundles of rank  $N$  on  $\mathbb{A}^2/\Gamma$ . In particular, if  $\mathfrak{g} = \mathfrak{sl}(k)$  it follows that

- 1) By H. Nakajima the geometry of vector bundles of rank  $n$  on  $\mathbb{A}^2/\Gamma_k$  is related to integrable modules over  $\mathfrak{sl}(k)_{\text{aff}}$  of level  $N$ .
- 2) By I. Frenkel's suggestion (and by our Conjecture 4.14) the geometry of vector bundles of rank  $N$  on  $\mathbb{A}^2/\Gamma_k$  is related to integrable modules over  $\mathfrak{sl}(N)_{\text{aff}}$  of level  $k$ .

On the other hand, in the representation theory of affine Lie algebras there is a well-known relation, due to I. Frenkel, between integrable modules over  $\mathfrak{sl}(k)_{\text{aff}}$  of level  $N$  and integrable modules over  $\mathfrak{sl}(N)_{\text{aff}}$  of level  $k$ . This connection is called *level-rank duality*; one of its aspects is discussed in [12]. It turns out that combining the results of [12] with the results of [24] one gets a proof of our main conjecture (in a slightly weaker form).

It is of course reasonable to ask why  $G$ -bundles on  $\mathbb{A}^2/\Gamma_k$  have anything to do with the sought-for affine Grassmannian of  $G_{\text{aff}}$ . We don't have a satisfactory answer to this question, though some sort of explanation (which would be too long to reproduce in the Introduction) is provided in Section 3. Also we have been informed by E. Witten that this phenomenon (at least when  $G$  is simply laced) has an explanation in terms of 6-dimensional conformal field theory (cf. [27]).

In [3] we explore other aspects of 1); in particular, we consider (mostly conjectural) affine analogs of convolution of  $G(\mathcal{O})$ -equivariant perverse sheaves on  $\text{Gr}_G$ , the analog of the so called Beilinson-Drinfeld Grassmannian, etc.

**1.4.** This paper is organized as follows: in Section 2 we review the basic results about the affine Grassmannian of a reductive group  $G$  that we would like to generalize to the affine case. In Section 3 we discuss how the affine Grassmannian of  $G_{\text{aff}}$  looks like from the naive (i.e. set-theoretic and not algebraic-geometric) point of view. In Section 4 we describe our affine analogs of the transversal slices  $\overline{\mathcal{W}}_{G,\mu}^\lambda$  and formulate the main conjectures about them (linking them to integrable representations of  $G_{\text{aff}}^\vee$ ). The next three sections are devoted to a verification of these conjectures in a number of special cases. In particular, in Section 7 we prove most of them in the case  $G = SL(N)$  and in Section 5 we prove all our conjectures when  $k$  is large compared to the other parameters and in Section 6 we prove almost all of our conjectures for  $k = 1$ .

**1.5. Acknowledgments.** As was mentioned above, the general idea saying that  $G$ -bundles on  $\mathbb{A}^2/\mathbb{Z}_k$  should have something to do with integrable modules of level  $k$  was explained to the first author by I. Frenkel, to whom we express our deepest gratitude. The authors would also like to thank B. Feigin, D. Gaitsgory, D. Kazhdan, H. Nakajima and E. Witten for many very interesting discussions on the subject. In addition our work was very much influenced by the paper [23] by I. Mirković and M. Vybornov to whom we also want to express our gratitude. A. B. was partially supported by the NSF Grant DMS-0600851. M. F. is grateful to MSRI for the hospitality and support; he was partially supported by the RFBR grant 09-01-00242 and the Science Foundation of the SU-HSE award No.09-08-0008 and 09-09-0009.

## 2. BASIC RESULTS ABOUT AFFINE GRASSMANNIAN

**2.1. Definition.** Let  $\mathcal{K} = \mathbb{C}((s))$ ,  $\mathcal{O} = \mathbb{C}[[s]]$ . By the *affine Grassmannian* of  $G$  we will mean the quotient  $\mathrm{Gr}_G = G(\mathcal{K})/G(\mathcal{O})$ . It is known (cf. [1, 22]) that  $\mathrm{Gr}_G$  is the set of  $\mathbb{C}$ -points of an ind-scheme over  $\mathbb{C}$ , which we will denote by the same symbol. Note that  $\mathrm{Gr}_G$  is defined for any (not necessarily reductive) group  $G$ .

Let  $\Lambda = \Lambda_G$  denote the coweight lattice of  $G$  and let  $\Lambda^\vee$  denote the dual lattice (this is the weight lattice of  $G$ ). We let  $2\rho_G^\vee$  denote the sum of the positive roots of  $G$ .

The group-scheme  $G(\mathcal{O})$  acts on  $\mathrm{Gr}_G$  on the left and its orbits can be described as follows. One can identify the lattice  $\Lambda_G$  with the quotient  $T(\mathcal{K})/T(\mathcal{O})$ . Fix  $\lambda \in \Lambda_G$  and let  $s^\lambda$  denote any lift of  $\lambda$  to  $T(\mathcal{K})$ . Let  $\mathrm{Gr}_G^\lambda$  denote the  $G(\mathcal{O})$ -orbit of  $s^\lambda$  (this is clearly independent of the choice of  $\lambda(s)$ ). The following result is well-known:

**Lemma 2.2.** (1)

$$\mathrm{Gr}_G = \bigcup_{\lambda \in \Lambda_G} \mathrm{Gr}_G^\lambda.$$

(2) We have  $\mathrm{Gr}_G^\lambda = \mathrm{Gr}_G^\mu$  if and only if  $\lambda$  and  $\mu$  belong to the same  $W$ -orbit on  $\Lambda_G$  (here  $W$  is the Weyl group of  $G$ ). In particular,

$$\mathrm{Gr}_G = \bigsqcup_{\lambda \in \Lambda_G^+} \mathrm{Gr}_G^\lambda.$$

(3) For every  $\lambda \in \Lambda^+$  the orbit  $\mathrm{Gr}_G^\lambda$  is finite-dimensional and its dimension is equal to  $\langle \lambda, 2\rho_G^\vee \rangle$ .

Let  $\overline{\mathrm{Gr}_G^\lambda}$  denote the closure of  $\mathrm{Gr}_G^\lambda$  in  $\mathrm{Gr}_G$ ; this is an irreducible projective algebraic variety; one has  $\mathrm{Gr}_G^\mu \subset \overline{\mathrm{Gr}_G^\lambda}$  if and only if  $\lambda - \mu$  is a sum of positive roots of  $G^\vee$ . We will denote by  $\mathrm{IC}^\lambda$  the intersection cohomology complex on  $\overline{\mathrm{Gr}_G^\lambda}$ . Let  $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$  denote the category of  $G(\mathcal{O})$ -equivariant perverse sheaves on  $\mathrm{Gr}_G$ . It is known that every object of this category is a direct sum of the  $\mathrm{IC}^\lambda$ 's.

The group  $\mathbb{C}^*$  acts naturally on  $G((s))$  by “loop rotation”; in other words, any  $a \in \mathbb{C}^*$  acts on  $g(s) \in G((s))$  by  $g(s) \mapsto g(as)$ . The fixed point variety  $(\mathrm{Gr}_G^\lambda)^{\mathbb{C}^*}$  is known to be isomorphic as a  $G$ -variety (via the action of  $G$  on  $(\mathrm{Gr}_G^\lambda)^{\mathbb{C}^*}$  coming from the natural identification  $G = G(\mathcal{O})^{\mathbb{C}^*}$ ) to  $G/P_\lambda$  where  $P_\lambda \subset G$  is the parabolic subgroup of  $G$  satisfying the following conditions:

- 1)  $P_\lambda$  contains the standard Borel subgroup  $B_+$ ;
- 2) The Weyl group of the corresponding Levi subgroup  $M_\lambda$  is equal to the stabilizer  $W_\lambda$  of  $\lambda$  in  $W$ .

In particular,  $(\mathrm{Gr}_G^\lambda)^{\mathbb{C}^*}$  is a projective variety. Moreover, the  $\mathbb{C}^*$ -action contracts all of  $\mathrm{Gr}_G^\lambda$  to  $(\mathrm{Gr}_G^\lambda)^{\mathbb{C}^*}$  and the corresponding map  $p_\lambda : \mathrm{Gr}_G^\lambda \rightarrow (\mathrm{Gr}_G^\lambda)^{\mathbb{C}^*}$  is a fiber bundle with fiber being isomorphic to an affine space [2].

The affine Grassmannian  $\mathrm{Gr}_G$  admits a canonical principal  $G$ -bundle  $\varepsilon : \widetilde{\mathrm{Gr}}_G \rightarrow \mathrm{Gr}_G$ . The set of  $\mathbb{C}$ -points of  $\widetilde{\mathrm{Gr}}_G$  is just  $G(\mathcal{K})/G(\mathcal{O})_1$  where  $G(\mathcal{O})_1$  is the kernel of the natural (“evaluation at 0”) map  $G(\mathcal{O}) \rightarrow G$ . This  $G$ -bundle is equivariant with respect to both  $G(\mathcal{K})$  and the loop rotations. The following lemma is easy and its proof is left to the reader:

**Lemma 2.3.** *Let  $x \in (\mathrm{Gr}_G)^{\mathbb{C}^*}$  (here as before the super-script  $\mathbb{C}^*$  means “fixed points of the loop rotation”). Then  $x \in (\mathrm{Gr}_G^\lambda)^{\mathbb{C}^*}$  if and only if the homomorphism  $\mathbb{C}^* \rightarrow G$  obtained via the action of  $\mathbb{C}^*$  in  $\varepsilon^{-1}(x)$  is conjugate to  $\lambda$ .*

**2.4. IC-stalks and the  $q$ -analog of the weight multiplicity.** Since every  $\mathrm{Gr}_G^\lambda$  is simply connected, it follows that  $\mathrm{IC}^\lambda|_{\mathrm{Gr}_G^\mu}$  is a constant complex, corresponding to some graded vector space  $\tilde{V}_\mu^\lambda = \oplus (\tilde{V}_\mu^\lambda)_i$ . We let  $V_\mu^\lambda = \tilde{V}_\mu^\lambda[-\langle \mu, 2\rho_G^\vee \rangle]$ . In other words,  $(V_\mu^\lambda)_i = (\tilde{V}_\mu^\lambda)_{i-\langle \mu, 2\rho_G^\vee \rangle}$ .

It follows from [21] and [8] (as well as from [13] whose proof is somewhat more adapted to our purposes) that the space  $V_\mu^\lambda$  can be described as follows.

Let  $L(\lambda)$  denote the finite-dimensional representation of  $G^\vee$  with highest weight  $\lambda$ . Let

$$\mathfrak{g}^\vee = \mathfrak{n}_+^\vee \oplus \mathfrak{t}^\vee \oplus \mathfrak{n}_-^\vee$$

be the natural triangular decomposition. Fix a regular nilpotent element  $e \in \mathfrak{n}_+^\vee$  and define an increasing filtration on  $F^i L(\lambda)$  by putting

$$x \in F^i L(\lambda) \Leftrightarrow e^i(x) = 0. \quad (2.1)$$

Thus we have the following

**Theorem 2.5.** 1) *The graded vector space  $V_\mu^\lambda$  lives only in even degrees.*

2) *We have a canonical isomorphism*

$$(V_\mu^\lambda)_{-2i} \simeq \mathrm{gr}_i^F L(\lambda)_\mu$$

(here  $L(\lambda)_\mu$  denotes the  $\mu$  weight space in  $L(\lambda)$ ).

**2.6. The  $q$ -analog of Kostant multiplicity formula.** The second assertion of Theorem 2.5 can be rewritten as follows. Introduce the generating functions:

$$\mathrm{IC}_\mu^\lambda(q) := \sum_i \dim(V_\mu^\lambda)_{-2i} q^i; \quad {}^e C_\mu^\lambda(q) := \sum_i \dim \mathrm{gr}_i^F L(\lambda)_\mu q^i.$$

Then the second assertion of Theorem 2.5 says that  $\mathrm{IC}_\mu^\lambda(q) = {}^e C_\mu^\lambda(q)$ . We shall refer to this function as *the  $q$ -character* of  $L(\lambda)_\mu$ .

The  $q$ -character of  $L(\lambda)_\mu$  can be given another purely combinatorial definition. Namely, let  $R$  denote the set of coroots of  $G$  (= roots of  $G^\vee$ ) and let  $R^+$  be the corresponding set of positive roots. Let  $\Lambda^{\mathrm{pos}}$  denote the subsemigroup of  $\Lambda$  consisting of elements of the form  $\sum n_j \alpha_j$  where  $n_j \in \mathbb{Z}_{\geq 0}$  and  $\alpha_j \in R^+$ . For all  $\beta \in \Lambda^{\mathrm{pos}}$  let us introduce the Kostant partition function of  $K_\beta(q)$  of  $\beta$  by:

$$\sum_{\beta \in \Lambda^{\mathrm{pos}}} K_\beta(q) := \prod_{\alpha \in R^+} (1 - qe^\alpha)^{-1},$$

For each  $\lambda \in \Lambda$  let us set  $w \cdot \lambda = w(\lambda + \rho_G) - \rho_G$ . Also let  $\ell : W \rightarrow \mathbb{Z}_+$  denote the length function.

Now we define

$$C_\mu^\lambda(q) := \sum_{w \in W} (-1)^{\ell(w)} K_{w \cdot \lambda - \mu}(q). \quad (2.2)$$

Then we have the following

**Theorem 2.7.** *For every  $\lambda, \mu \in \Lambda^+$  We have  $\mathrm{IC}_\mu^\lambda(q) = {}^e C_\mu^\lambda(q) = C_\mu^\lambda(q)$ .*

The equality  $\mathrm{IC}_\mu^\lambda(q) = C_\mu^\lambda(q)$  has been proved in Lusztig's paper [21] where the author also formulated the equality  ${}^e C_\mu^\lambda(q) = C_\mu^\lambda(q)$  as a conjecture. Note that this conjecture is purely representation-theoretic (i.e. it has nothing to do with the geometry of  $\mathrm{Gr}_G$ ); it was proved later by Brylinski [8]. Note however that the equality  $\mathrm{IC}_\mu^\lambda(q) = {}^e C_\mu^\lambda(q)$  has an independent proof (cf. [13] in which an isomorphism as in Theorem 2.5 is constructed). Note also that both  ${}^e C_\mu^\lambda(q)$  and  $C_\mu^\lambda(q)$  make sense when  $\mu$  is arbitrary (i.e. not necessarily). In this case it was shown in [15] that the equality  ${}^e C_\mu^\lambda(q) = C_\mu^\lambda(q)$  still holds up to certain power of  $q$  which depends on how “non-dominant”  $\mu$  is (cf. Theorem 7.6 in [15]).

**2.8. Transversal slices.** Consider the group  $G[s^{-1}] \subset G((s))$ ; let us denote by  $G[s^{-1}]_1$  the kernel of the natural (“evaluation at  $\infty$ ”) homomorphism  $G[s^{-1}] \rightarrow G$ . For any  $\lambda \in \Lambda$  let  $\mathrm{Gr}_{G,\lambda} = G[s^{-1}] \cdot s^\lambda$ . Then it is easy to see that one has

$$\mathrm{Gr}_G = \bigsqcup_{\lambda \in \Lambda^+} \mathrm{Gr}_{G,\lambda}$$

Let also  $\mathcal{W}_{G,\lambda}$  denote the  $G[s^{-1}]_1$ -orbit of  $s^\lambda$ . For any  $\lambda, \mu \in \Lambda^+$ ,  $\lambda \geq \mu$  set

$$\mathrm{Gr}_{G,\mu}^\lambda = \mathrm{Gr}_G^\lambda \cap \mathrm{Gr}_{G,\mu}, \quad \overline{\mathrm{Gr}}_{G,\mu}^\lambda = \overline{\mathrm{Gr}}_G^\lambda \cap \mathrm{Gr}_{G,\mu}$$

and

$$\mathcal{W}_{G,\mu}^\lambda = \mathrm{Gr}_G^\lambda \cap \mathcal{W}_{G,\mu}, \quad \overline{\mathcal{W}}_{G,\mu}^\lambda = \overline{\mathrm{Gr}}_G^\lambda \cap \mathcal{W}_{G,\mu}.$$

Note that  $\overline{\mathcal{W}}_{G,\mu}^\lambda$  contains the point  $s^\mu$  in it.

**Lemma 2.9.** (1) *The point  $s^\mu$  is the only  $\mathbb{C}^*$ -fixed point in  $\overline{\mathcal{W}}_{G,\mu}^\lambda$ . The action of  $\mathbb{C}^*$  on  $\overline{\mathcal{W}}_{G,\mu}^\lambda$  is “repelling”, i.e. for any  $w \in \overline{\mathcal{W}}_{G,\mu}^\lambda$  we have*

$$\lim_{a \rightarrow \infty} a(w) = s^\mu.$$

(2) *The variety  $\overline{\mathrm{Gr}}_{G,\mu}^\lambda$  is a fiber bundle over  $G/P_\mu$  with fiber  $\overline{\mathcal{W}}_{G,\mu}^\lambda$ .*

(3) *There exists an open subset  $U$  in  $\mathrm{Gr}_G^\mu$  and an open embedding  $U \times \overline{\mathcal{W}}_{G,\mu}^\lambda \hookrightarrow \overline{\mathrm{Gr}}_G^\lambda$  such that the diagram*

$$\begin{array}{ccc} U \times \{s^\mu\} & \longrightarrow & \mathrm{Gr}_G^\mu \times \{s^\mu\} \\ \downarrow & & \downarrow \\ U \times \overline{\mathcal{W}}_{G,\mu}^\lambda & \longrightarrow & \overline{\mathrm{Gr}}_G^\lambda \end{array}$$

*is commutative. In other words,  $\overline{\mathcal{W}}_{G,\mu}^\lambda$  is a transversal slice to  $\mathrm{Gr}_G^\mu$  inside  $\overline{\mathrm{Gr}}_G^\lambda$ . In particular, the stalk of  $\mathrm{IC}_{\overline{\mathcal{W}}_{G,\mu}^\lambda}$  at the point  $s^\mu$  is equal to  $V_\mu^\lambda[-\langle \mu, 2\rho^\vee \rangle]$  (note that  $\langle \mu, 2\rho^\vee \rangle$  is the dimension on  $\mathrm{Gr}_G^\mu$ ).*

*Proof.* The first two statements are obvious, and the third one follows from Propositions 1.3.1 and 1.3.2 of [16].  $\square$

### 3. THE DOUBLE AFFINE GRASSMANNIAN: THE NAIVE APPROACH

**3.1. The affine Kac-Moody group and its Langlands dual.** In this paper for convenience we adopt the polynomial version of loop groups (as opposed to formal loops version).

Let  $G$  be a connected simply connected algebraic group with simple Lie algebra  $\mathfrak{g}$ . Consider the corresponding polynomial algebra  $\mathfrak{g}[t, t^{-1}]$  and the group  $G[t, t^{-1}]$ . It is well-known that  $G[t, t^{-1}]$  has a canonical central extension  $\widehat{G}$ :

$$1 \rightarrow \mathbb{C}^* \rightarrow \widehat{G} \rightarrow G[t, t^{-1}] \rightarrow 1.$$

We denote by  $\widehat{\mathfrak{g}}$  the corresponding Lie algebra; it fits into the exact sequence

$$0 \rightarrow \mathbb{C} \rightarrow \widehat{\mathfrak{g}} \rightarrow \mathfrak{g}[t, t^{-1}] \rightarrow 0.$$

This central extension corresponds to a choice of an invariant symmetric bilinear form  $(\cdot, \cdot)$  on  $\mathfrak{g}$  which is integral and even on  $\Lambda \subset \mathfrak{t} \subset \mathfrak{g}$ . We shall assume that  $(\cdot, \cdot)$  is the minimal possible such form (i.e. that  $(\alpha, \alpha) = 2$  for any short coroot  $\alpha$  of  $G$ ).

The group  $\mathbb{C}^*$  acts naturally on  $G[t, t^{-1}]$  and this action lifts to  $\widehat{G}$ . We denote the corresponding semi-direct product by  $G_{\text{aff}}$ ; we also let  $\mathfrak{g}_{\text{aff}}$  denote its Lie algebra. Thus  $\mathfrak{g}_{\text{aff}}$  is an untwisted affine Kac-Moody Lie algebra, in particular, it can be described by the corresponding affine root system. We let  $G'_{\text{aff}}$  be the quotient of  $G_{\text{aff}}$  by the central  $\mathbb{C}^*$ .

Let now  $\mathfrak{g}_{\text{aff}}^{\vee}$  denote the *Langlands dual* affine Lie algebra. By definition, this is an affine Kac-Moody Lie algebra whose root system is dual to that of  $\mathfrak{g}_{\text{aff}}$ . Note that in general (when  $\mathfrak{g}$  is not simply laced) the algebra  $\mathfrak{g}_{\text{aff}}^{\vee}$  is not isomorphic to  $(\mathfrak{g}^{\vee})_{\text{aff}}$  (here  $\mathfrak{g}^{\vee}$  as before denotes the Langlands dual Lie algebra of  $\mathfrak{g}$ ). Moreover, if  $\mathfrak{g}$  is not simply laced, then  $\mathfrak{g}_{\text{aff}}^{\vee}$  is a twisted affine Lie algebra. However, the algebra  $\mathfrak{g}_{\text{aff}}^{\vee}$  always contains  $\mathfrak{g}^{\vee} \times \mathbb{C}^2$  as a Levi subalgebra.

Let also  $G_{\text{aff}}^{\vee}$  be the corresponding Langlands dual group. This is a connected affine Kac-Moody group in the sense of [18] which is uniquely characterized by the following two properties:

- The Lie algebra of  $G_{\text{aff}}^{\vee}$  is  $\mathfrak{g}_{\text{aff}}^{\vee}$ ;
- The embedding  $\mathfrak{g}^{\vee} \hookrightarrow \mathfrak{g}_{\text{aff}}^{\vee}$  lifts to an embedding  $G^{\vee} \hookrightarrow G_{\text{aff}}^{\vee}$ .

The group  $G_{\text{aff}}^{\vee}$  maps naturally to  $\mathbb{C}^*$  (this homomorphism is dual to the central embedding  $\mathbb{C}^* \rightarrow G_{\text{aff}}$ ). We denote the kernel of this homomorphism by  $\widehat{G}^{\vee}$  and we let  $\widehat{\mathfrak{g}}^{\vee}$  denote its Lie algebra.

**3.2. The weight lattice of  $G_{\text{aff}}^{\vee}$ .** In what follows we shall write just  $\Lambda$  instead of  $\Lambda_G$ ; also from now on we shall usually denote elements of  $\Lambda$  by  $\bar{\lambda}, \bar{\mu} \dots$  (instead of just writing  $\lambda, \mu \dots$  in order to distinguish them from the coweights of  $G_{\text{aff}}$  (= weights of  $G_{\text{aff}}^{\vee}$ ).

The weight lattice of  $G_{\text{aff}}^{\vee}$  (resp. of  $\widehat{G}^{\vee}$ ) is naturally identified with  $\Lambda_{\text{aff}} = \mathbb{Z} \times \Lambda \times \mathbb{Z}$  (resp. with  $\widehat{\Lambda} = \mathbb{Z} \times \Lambda$ ). Here the first  $\mathbb{Z}$ -factor is responsible for the center of  $G_{\text{aff}}^{\vee}$  (or  $\widehat{G}^{\vee}$ ); it can also be thought of as coming from the loop rotation in  $G_{\text{aff}}$ . The second  $\mathbb{Z}$ -factor is responsible for the loop rotation in  $G_{\text{aff}}^{\vee}$  (and thus it is absent in the case of  $\widehat{G}^{\vee}$ ; it may also be thought of as coming from the center of  $G_{\text{aff}}$ ). We denote by  $\Lambda_{\text{aff}}^+$  the set of dominant weights of  $G_{\text{aff}}^{\vee}$  (which is the same as the set of dominant coweights of  $G_{\text{aff}}$ ). We also denote by  $\Lambda_{\text{aff}, k}$  the set of weights of  $G_{\text{aff}}^{\vee}$  of level  $k$ , i.e. all the weights of the form  $(k, \bar{\lambda}, n)$ . We put  $\Lambda_{\text{aff}, k}^+ = \Lambda_{\text{aff}}^+ \cap \Lambda_{\text{aff}, k}$ .



Let  $\Lambda_k^+ \subset \Lambda$  denote the set of dominant coweights of  $G$  such that  $\langle \bar{\lambda}, \alpha \rangle \leq k$  when  $\alpha$  is the highest root of  $\mathfrak{g}$ . Then it is well-known that a weight  $(k, \bar{\lambda}, n)$  of  $G_{\text{aff}}^\vee$  lies in  $\Lambda_{\text{aff},k}^+$  if and only if  $\bar{\lambda} \in \Lambda_k^+$  (thus  $\Lambda_{\text{aff},k} = \Lambda_k^+ \times \mathbb{Z}$ ).

Let also  $W_{\text{aff}}$  denote affine Weyl group of  $G$  which is the semi-direct product of  $W$  and  $\Lambda$ . It acts on the lattice  $\Lambda_{\text{aff}}$  (resp.  $\hat{\Lambda}$ ) preserving each  $\Lambda_{\text{aff},k}$  (resp. each  $\hat{\Lambda}_k$ ). In order to describe this action explicitly it is convenient to set  $W_{\text{aff},k} = W \ltimes k\Lambda$  which naturally acts on  $\Lambda$ . Of course the groups  $W_{\text{aff},k}$  are canonically isomorphic to  $W_{\text{aff}}$  for all  $k$ . Then the restriction of the  $W_{\text{aff}}$ -action to  $\Lambda_{\text{aff},k} \simeq \Lambda \times \mathbb{Z}$  comes from the natural  $W_{\text{aff},k}$ -action on the first multiple.

It is well known that every  $W_{\text{aff}}$ -orbit on  $\Lambda_{\text{aff},k}$  contains unique element of  $\Lambda_{\text{aff},k}^+$ . This is equivalent to saying that  $\Lambda_k^+ \simeq \Lambda/W_{\text{aff},k}$ .

It is clear that for two integers  $0 \leq k < l$  we have  $\Lambda_k^+ \subset \Lambda_l^+$ . However, if  $k|l$ , we also have the natural embedding  $W_{\text{aff},l} \subset W_{\text{aff},k}$  and thus the identification  $\Lambda_k^+ = \Lambda/W_{\text{aff},k}$ ,  $\Lambda_l^+ = \Lambda/W_{\text{aff},l}$  gives rise to the projection  $\Lambda_l^+ \rightarrow \Lambda_k^+$  which is equal to identity on  $\Lambda_k^+$ .

Let  $\Gamma_k$  denote the group of roots of unity of order  $k$ . Then we claim the following:

**Lemma 3.3.** *There is a natural bijection between  $\Lambda/W_{\text{aff},k}$  and the set of conjugacy classes of homomorphisms  $\Gamma_k \rightarrow G$ .<sup>3</sup>*

*Proof.* Recall that  $\Lambda = \text{Hom}(\mathbb{G}_m, T)$ . Thus, since  $\Gamma_k$  is a closed subscheme of  $\mathbb{G}_m$ , given every  $\bar{\lambda} \in \Lambda$  we may restrict it to  $\Gamma_k$  and get a homomorphism  $\Gamma_k \rightarrow T$ . By composing it with the embedding  $T \hookrightarrow G$  we get a homomorphism  $\Gamma_k \rightarrow G$  which clearly depends only on the image of  $\lambda$  in  $\Lambda/W_{\text{aff},k}$ . Thus we get a well-defined map  $\Lambda/W_{\text{aff},k} \rightarrow (\text{Hom}(\Gamma_k, G)/G)$ . The surjectivity of this map follows from the fact that  $\Gamma_k$  is finite and thus diagonalizable. For the injectivity, note that any two elements in  $T$  which are conjugate in  $G$  lie in the same  $W$ -orbit in  $T$ . Thus it is enough to show that for any two homomorphisms  $\lambda, \mu : \mathbb{G}_m \rightarrow T$  whose restrictions to  $\Gamma_k$  coincide the difference  $\lambda - \mu$  is divisible by  $k$ . This is enough to check for  $T = \mathbb{G}_m$  where it is obvious.  $\square$

**3.4. Representations of  $G_{\text{aff}}^\vee$ .** In the sequel we shall be concerned with representations of  $G_{\text{aff}}^\vee$  (i.e. with integrable representations of the Lie algebra  $\mathfrak{g}_{\text{aff}}^\vee$  which integrate to  $G_{\text{aff}}^\vee$ ). The following results are well-known (cf. [18]):

- a) The irreducible representations of  $\hat{G}^\vee$  of level  $k$  are in one-to-one correspondence with elements of  $\Lambda_k^+$ ;
- b) The irreducible representations of  $G_{\text{aff}}^\vee$  of level  $k$  are in one-to-one correspondence with elements of  $\mathbb{Z} \times \Lambda_k^+$ ;
- c) For any irreducible representation  $L$  of  $G_{\text{aff}}^\vee$ , its restriction to  $\hat{G}^\vee$  is irreducible.

For every  $\lambda \in \Lambda_{\text{aff},k}^+$  we denote by  $L(\lambda)$  the corresponding representation of  $G_{\text{aff}}^\vee$ . For every  $\mu \in \Lambda_{k,\text{aff}}$  we denote by  $L(\lambda)_\mu$  the  $\mu$ -weight space of  $L(\lambda)$ . It is known to be finite-dimensional.

Recall that the Lie algebra  $\mathfrak{g}_{\text{aff}}^\vee$  has a triangular decomposition

$$\mathfrak{g}_{\text{aff}}^\vee = \mathfrak{n}_{-,\text{aff}}^\vee \oplus \mathfrak{t}_{\text{aff}}^\vee \oplus \mathfrak{n}_{+,\text{aff}}^\vee.$$

---

<sup>3</sup>This lemma is true for an arbitrary reductive group  $G$  (with the same proof). However, the identification  $\Lambda_k^+ = \Lambda/W_{\text{aff},k}$  is only true when  $G$  is simply connected.

The Lie algebra  $\mathfrak{n}_{+, \text{aff}}^\vee$  is generated by the standard Cartan generators  $e_0, e_1, \dots, e_r$ . Set  $e = e_0 + \dots + e_r$ . This is an affine analogue of a regular nilpotent element considered in Section 2. We define a filtration  $F^i L(\lambda)$  in the same way as in (2.1); it can be restricted to every weight space  $L(\lambda)_\mu$ . We set  $\text{gr}^F L(\lambda)_\mu$  to be the associated graded space.

**3.5. The affine Grassmannian of  $G_{\text{aff}}$ : the naive approach.** The rest of this Section is written for motivational purposes only; it will not be formally used in the rest of the paper.

Let  $\mathcal{K} = \mathbb{C}[s, s^{-1}]$ ,  $\mathcal{O} = \mathbb{C}[s]$  (note that this notation is different from the one we used in Section 2.1).

Then we may consider the group  $G_{\text{aff}}(\mathcal{K})$  and its subgroup  $G_{\text{aff}}(\mathcal{O})$ .

Let  $\pi$  denote the natural map from  $G_{\text{aff}}(\mathcal{K})$  to  $\mathcal{K}^* = \mathbb{C}^* \times \mathbb{Z}$ . We denote by  $G_{\text{aff}}^+(\mathcal{K})$  the sub-semigroup of  $G_{\text{aff}}(\mathcal{K})$  defined as follows:

$$G_{\text{aff}}^+(\mathcal{K}) = G_{\text{aff}}(\mathcal{O}) \cup \{g \in G_{\text{aff}}(\mathcal{K}) \text{ such that } \pi(g) = as^k \text{ where } k \geq 0\}.$$

For each  $k \geq 0$  we consider the subset  $G_{\text{aff},k}(\mathcal{K})$  of  $G_{\text{aff}}^+(\mathcal{K})$  consisting of those  $g \in G_{\text{aff}}^+(\mathcal{K})$  for which  $\pi(g) = as^k$  (note that for all  $k \neq 0$  we may replace  $G_{\text{aff}}^+$  by  $G_{\text{aff}}$  here).

The same definitions make sense when  $G_{\text{aff}}$  is replaced by  $G'_{\text{aff}}$ .

**Lemma 3.6.** *The set  $G_{\text{aff}}(\mathcal{O}) \backslash G_{\text{aff},k}(\mathcal{K}) / G_{\text{aff}}(\mathcal{O})$  is in natural bijection with  $\Lambda_{\text{aff},k}^+$ . Similarly,  $G'_{\text{aff},k}(\mathcal{O}) \backslash G'_{\text{aff},k}(\mathcal{K}) / G'_{\text{aff}}(\mathcal{O})$  is in natural bijection with  $\Lambda_k^+$ .<sup>4</sup>*

Let us formulate Lemma 3.6 a bit more precisely. Recall that  $G_{\text{aff}}$  contains  $\mathbb{G}_m \times G \times \mathbb{G}_m$  as a subgroup (here the first  $\mathbb{G}_m$ -factor stands for the center and the second one — for the loop rotation). In particular,  $G_{\text{aff}}(\mathcal{K})$  contains  $\mathcal{K}^* \times G(\mathcal{K}) \times \mathcal{K}^*$ . For every  $\lambda \in \Lambda_{\text{aff},k}$  of the form  $(k, \bar{\lambda}, n)$  we may consider the element  $s^\lambda = (s^n, s^{\bar{\lambda}}, s^k)$  in  $\mathcal{K}^* \times T(\mathcal{K}) \times \mathcal{K}^* \subset G_{\text{aff},k}(\mathcal{K})$ .

We claim that

- (1)  $G_{\text{aff},k}(\mathcal{K}) = \bigcup_{\lambda \in \Lambda_{\text{aff},k}} G_{\text{aff}}(\mathcal{O}) \cdot s^\lambda \cdot G_{\text{aff}}(\mathcal{O})$ ;
- (2) We have  $G_{\text{aff}}(\mathcal{O}) \cdot s^\lambda \cdot G_{\text{aff}}(\mathcal{O}) = G_{\text{aff}}(\mathcal{O}) \cdot s^{\lambda'} \cdot G_{\text{aff}}(\mathcal{O})$  if and only if  $\lambda$  and  $\lambda'$  are in the same orbit of  $W_{\text{aff}}$ .

Similar statements hold for  $G'_{\text{aff}}$  instead of  $G_{\text{aff}}$ .

This is a direct “affine” analog of the first two assertions of Lemma 2.2 and for  $k = 0$  it follows from there. We shall be mostly concerned with the case  $k > 0$ . In this case one can prove the above two assertions directly; however, for the future it will be more instructive to prove it using the following trick. We shall do it for  $G'_{\text{aff}}$  instead of  $G_{\text{aff}}$  (it is easy to see that our statements for  $G_{\text{aff}}$  and  $G'_{\text{aff}}$  are equivalent).

Let  $\bar{S}_k = \text{Spec } \mathbb{C}[x, y, z]/xy - z^k$  and let  $S_k$  be the complement to the point  $(0, 0, 0)$  in  $\bar{S}_k$ . This is a smooth surface.

The natural map  $p_k : \mathbb{A}^2 \rightarrow \bar{S}_k$  given by

$$(u, v) \mapsto (u^k, v^k, uv)$$

---

<sup>4</sup>If one uses  $\Lambda_{\text{aff},k}/W_{\text{aff}}$  and  $\Lambda/W_{\text{aff},k}$  instead  $\Lambda_{\text{aff},k}^+$  and  $\Lambda_k^+$  the statement becomes true for arbitrary reductive  $G$ .

identifies  $S_k$  with  $(\mathbb{A}^2 \setminus \{0\})/\Gamma_k$ , where  $\Gamma_k$  acts on  $\mathbb{A}^2$  by  $\zeta(u, v) = (\zeta v, \zeta^{-1}v)$  (note that the action of  $\Gamma_k$  on  $(\mathbb{A}^2 \setminus \{0\})$  is free).

For any  $\mathbb{C}$ -variety  $S$  we shall denote by  $\text{Bun}_G(S)$  the set of isomorphism classes of principal  $G$ -bundles on  $S$ .

**Proposition 3.7.** *The following sets are in natural bijection:*

- (1) *The set  $\text{Bun}_G(S_k)$ ;*
- (2) *The set  $G'_{\text{aff}}(\mathcal{O}) \backslash G'_{\text{aff},k}/G'_{\text{aff}}(\mathcal{O})$ ;*
- (3) *The set of  $G$ -conjugacy classes of homomorphisms  $\Gamma_k \rightarrow G$ .*

*Proof.* Let us first establish the bijection between (1) and (3). Let  $\mathcal{F}$  denote a  $G$ -bundle on  $S_k$ . Consider the  $G$ -bundle  $p_k^*(\mathcal{F})$  on  $\mathbb{A}^2 \setminus \{0\}$ . It extends uniquely to the whole of  $\mathbb{A}^2$  and thus it is trivial. On the other hand, the bundle  $p_k^*(\mathcal{F})$  is  $\Gamma_k$ -equivariant. Since this bundle is trivial, such an equivariant structure gives rise to a homomorphism  $\Gamma_k \rightarrow G$  defined uniquely up to conjugacy. This defines a map  $\text{Bun}_G(S_k) \rightarrow \text{Hom}(\Gamma_k, G)/G$ . It is clear that this is actually a bijection, since a  $\Gamma_k$ -equivariant  $G$ -bundle on  $\mathbb{A}^2 \setminus \{0\}$  descends uniquely to  $S_k$ .

Let us now establish the bijection between (1) and (2). To do that let us denote by  $\sigma$  the automorphism of  $G(\mathcal{K})$  sending  $g(t, s)$  to  $g(ts, s)$ . Let us now identify  $\pi^{-1}(s^k)$  with  $G(\mathcal{K})$  with right  $G(\mathcal{K})$  action being the standard one (by right shifts) and with left  $G(\mathcal{K})$ -action given by  $g(h) = \sigma^k(g)h$ . Thus we have the bijection

$$G'_{\text{aff}}(\mathcal{O}) \backslash G'_{\text{aff},k}/G'_{\text{aff}}(\mathcal{O}) = G[ts^k, t^{-1}s^{-k}, s] \backslash G[t, t^{-1}, s, s^{-1}]/G[t, t^{-1}, s]. \quad (3.1)$$

Let  $U_k = \text{Spec } \mathbb{C}[ts^k, t^{-1}s^{-k}, s]$ ,  $V_k = \text{Spec } \mathbb{C}[t, t^{-1}, s]$ . Both  $U_k$  and  $V_k$  are isomorphic to  $\mathbb{G}_m \times \mathbb{A}^1$  and thus every  $G$ -bundle on either of these surfaces is trivial. Both  $U_k$  and  $V_k$  contain  $W = \text{Spec } \mathbb{C}[t, t^{-1}, s, s^{-1}] \simeq \mathbb{G}_m \times \mathbb{G}_m$  as a Zariski open subset. Thus the RHS of (3.1) can be identified with  $\text{Bun}_G(S'_k)$  where  $S'_k$  is obtained by gluing  $U_k$  and  $V_k$  along  $W$ . Hence it remains to construct an isomorphism  $S'_k \xrightarrow{\sim} S_k$ . Such an isomorphism can be obtained by setting  $x = ts^k, y = t^{-1}, z = s$ .  $\square$

The affine Grassmannian  $G_{\text{aff}}(\mathcal{K})/G_{\text{aff}}(\mathcal{O})$  of  $G_{\text{aff}}$  is a  $\mathbb{Z}$ -torsor over the corresponding quotient for  $G'_{\text{aff}}$ . It is not clear how to think about the affine Grassmannian of  $G_{\text{aff}}$  in terms of algebraic geometry. Let us note though that for each  $k > 0$  the set of double cosets  $G_{\text{aff}}(\mathcal{O}) \backslash G_{\text{aff},k}(\mathcal{K})/G_{\text{aff}}(\mathcal{O})$  is also a  $\mathbb{Z}$ -torsor over the set  $G'_{\text{aff}}(\mathcal{O}) \backslash G'_{\text{aff},k}(\mathcal{K})/G'_{\text{aff}}(\mathcal{O}) = \Lambda_k^+$ . Moreover, by analyzing the geometric construction of this torsor given in [7] more carefully and using the embedding  $\mathbb{A}^2 \subset \mathbb{P}^2$  one can show it follows from the above that the above  $\mathbb{Z}$ -torsor can be canonically trivialized. Thus, if we set  $\text{Gr}_{G_{\text{aff},k}} = G_{\text{aff},k}(\mathcal{K})/G_{\text{aff}}(\mathcal{O})$ , we see that  $G_{\text{aff}}(\mathcal{O})$ -orbits on  $\text{Gr}_{G_{\text{aff},k}}$  are parameterized by  $\Lambda_k^+ \times \mathbb{Z} = \Lambda_{\text{aff},k}^+$ . Alternatively,  $G_{\text{aff}}(\mathcal{O})$ -orbits on  $\text{Gr}_{G_{\text{aff},k}}$  are parameterized by  $\text{Bun}_G(S_k) \times \mathbb{Z}$ . This remark might serve as a motivation for the constructions of next Section.

#### 4. THE MAIN CONSTRUCTION

**4.1. The moduli space of  $G$ -bundles on  $\mathbb{A}^2$ .** Here we follow the exposition of [4]. Let  $\mathbf{S}$  be a smooth projective surface containing  $\mathbb{A}^2$  as an open subset and let  $\mathbf{D}_\infty$  denote the complement of  $\mathbb{A}^2$  in  $\mathbf{S}$  (we shall refer to it as “the divisor at  $\infty$ ”). In what follows  $\mathbf{S}$  will almost always be  $\mathbb{P}^2$ , or sometimes  $\mathbb{P}^1 \times \mathbb{P}^1$ . For an integer  $a$  let  $\text{Bun}_G^a(\mathbb{A}^2)$  denote the moduli

space of principal  $G$ -bundles on  $\mathbf{S}$  of second Chern class  $a$  with a chosen trivialization on  $\mathbf{D}_\infty$  (cf. [4] for the discussion of the notion of  $c_2$  in this case). It is easy to see that  $\text{Bun}_G^d$  does not depend on the choice of  $\mathbf{S}$ . It is shown in [4] that this space has the following properties:

- a)  $\text{Bun}_G^a(\mathbb{A}^2)$  is non-empty if and only if  $a \geq 0$ ;
- b) For  $a \geq 0$  the space  $\text{Bun}_G^a(\mathbb{A}^2)$  is an irreducible smooth quasi-affine variety of dimension  $2a\check{h}$  where  $\check{h}$  denotes the dual Coxeter number of  $G$ .

**4.2. The Uhlenbeck space of  $\mathbb{A}^2$ .** In [4] we construct an affine scheme  $\mathcal{U}_G^a(\mathbb{A}^2)$  containing  $\text{Bun}_G^a(\mathbb{A}^2)$  as a dense open subset which we are going to call the *Uhlenbeck space* of bundles on  $\mathbb{A}^2$ .<sup>5</sup>

The scheme  $\mathcal{U}_G^a(\mathbb{A}^2)$  is still irreducible but in general it is highly singular. The main property of  $\mathcal{U}_G^a(\mathbb{A}^2)$  is that it possesses the following stratification:

$$\text{Bun}_G^a(\mathbb{A}^2) = \bigcup_{0 \leq b \leq a} \text{Bun}_G^b(\mathbb{A}^2) \times \text{Sym}^{a-b}(\mathbb{A}^2). \quad (4.1)$$

Here each  $\text{Bun}_G^b(\mathbb{A}^2) \times \text{Sym}^{a-b}(\mathbb{A}^2)$  is a locally closed subset of  $\mathcal{U}_G^a(\mathbb{A}^2)$  and its closure is equal to the union of similar subsets corresponding to all  $b' \leq b$ .

We shall denote by  $\text{Bun}_G(\mathbb{A}^2)$  (resp.  $\mathcal{U}_G(\mathbb{A}^2)$ ) the disjoint union of all  $\text{Bun}_G^a(\mathbb{A}^2)$  (resp. of  $\mathcal{U}_G^a(\mathbb{A}^2)$ ).

Let us note that the group  $G \times \text{GL}(2)$  acts naturally on  $\text{Bun}_G^a(\mathbb{A}^2)$ : here the first factor acts by changing the trivialization on  $\mathbf{D}_\infty$  and the second factor acts on  $\mathbb{A}^2$  (formally we should take  $\mathbf{S} = \mathbb{P}^2$  for this; then  $\text{GL}(2)$  acts on  $\mathbf{S}$  preserving  $\mathbf{D}_\infty$  and thus acts on  $\text{Bun}_G^a(\mathbb{A}^2)$ ). It is easy to deduce from the construction of [4] that this action extends to an action of the same group on the Uhlenbeck space  $\mathcal{U}_G^a(\mathbb{A}^2)$ .

The following remark (also proved in [4]) will be needed in the future: consider the central  $\mathbb{C}^* \subset \text{GL}(2)$ . Then it acts on  $\mathcal{U}_G^a(\mathbb{A}^2)$  with unique fixed point  $0^a \in \mathcal{U}_G^a(\mathbb{A}^2)$ . In terms of the stratification (4.1) the point  $0^a$  corresponds to the case  $b = 0$  (recall that  $\text{Bun}_G^0(\mathbb{A}^2)$  consists of just one point (the trivial bundle)) and to the point  $(0, 0)$  taken with multiplicity  $a$  (as a point in  $\text{Sym}^a(\mathbb{A}^2)$ ). Moreover, the above  $\mathbb{C}^*$  action contracts all of  $\mathcal{U}_G^a(\mathbb{A}^2)$  to  $0^a$  (i.e. for any  $p \in \mathcal{U}_G^a(\mathbb{A}^2)$  we have  $\lim_{z \rightarrow 0} z(p) = 0^a$ ). In particular, any closed  $\mathbb{C}^*$ -invariant subvariety  $W$  of  $\mathcal{U}_G^a(\mathbb{A}^2)$  must also contain the point  $0^a$  and all of  $W$  is contracted to  $0^a$  by the  $\mathbb{C}^*$ -action.

**4.3. The factorization morphism.** Let  $\mathbb{A}^{(a)}$  denote the  $a$ -th symmetric power of  $\mathbb{A}^1$ . This is the space of all effective divisors on  $\mathbb{A}^1$  of degree  $a$ . As a scheme, it is, of course, isomorphic to the affine space  $\mathbb{A}^a$ .

In [4] we construct the *factorization morphism*

$$\pi^a : \text{Bun}_G^a(\mathbb{A}^2) \rightarrow \mathbb{A}^{(a)}.$$

This morphism, in fact, depends on the choice of a direction in  $\mathbb{A}^2$ . Below we are going to work with the “horizontal” factorization morphism (in the terminology of [4]), i.e. our  $\mathbb{A}^{(a)}$  should be thought of as the  $a$ -th symmetric power of the horizontal axis in  $\mathbb{A}^2$ . To describe

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<sup>5</sup>In fact in [4] we give several definitions of  $\mathcal{U}_G^a(\mathbb{A}^2)$  about which we only know that the corresponding reduced schemes coincide. In this paper we shall take  $\mathcal{U}_G^a(\mathbb{A}^2)$  to be reduced by definition.

this morphism, it is convenient to take  $\mathbf{S} = \mathbb{P}^1 \times \mathbb{P}^1$ . Let  $\mathcal{F} \in \text{Bun}_G^a(\mathbb{A}^2)$ . Then  $x \in \mathbb{A}^1$  lies in the support of the divisor  $\pi^a(\mathcal{F})$  if and only if the restriction of  $\mathcal{F}$  to the line  $\{a\} \times \mathbb{P}^1$  is non-trivial (i.e.  $\{a\} \times \mathbb{P}^1$  is a “jumping line” for  $\mathcal{F}$ ). We refer the reader to [4] for the formal definition of  $\pi^a$ .

The term “factorization morphism” comes from the following property of  $\pi^a$ : assume that we are given two effective divisors  $D_1$  and  $D_2$  on  $\mathbb{A}^1$  of degrees  $a_1$  and  $a_2$ . Assume that  $D_1$  and  $D_2$  are disjoint; in other words let assume that the supports of  $D_1$  and  $D_2$  do not intersect. Let  $a = a_1 + a_2$ . Then  $\pi^a$  enjoys the following *factorization property*:

$$\text{There a canonical isomorphism } (\pi^a)^{-1}(D_1 + D_2) = (\pi^{a_1})^{-1}(D_1) \times (\pi^{a_2})^{-1}(D_2). \quad (4.2)$$

The factorization morphism  $\pi^a$  extends to  $\mathcal{U}_G^a(\mathbb{A}^2)$ ; abusing the notation we shall denote it by the same symbol. The factorization property (4.2) holds for this new  $\pi^a$  as well.

**4.4. Action of  $\Gamma_k$ .** Let  $\Gamma_k$  as before denote the group of roots of unity of order  $k$ . We consider the embedding  $\Gamma_k \hookrightarrow (\mathbb{C}^*)^2 \subset \text{GL}(2)$  sending every  $\zeta \in \Gamma_k$  to  $(\zeta, \zeta^{-1})$ . We shall refer to the corresponding action of  $\Gamma_k$  on  $\mathbb{A}^2$  as the *symplectic action* (since it preserves the natural symplectic form on  $\mathbb{A}^2$ ). In what follows we shall always take  $\mathbf{S}$  to be either  $\mathbb{P}^2$  of  $\mathbb{P}^1 \times \mathbb{P}^1$ ; in both cases the symplectic action of  $\Gamma_k$  on  $\mathbb{A}^2$  extends to an action on  $\mathbf{S}$ .

Since the group  $G$  acts on  $\text{Bun}_G(\mathbb{A}^2)$  (cf. previous subsection) it follows that a choice of  $\bar{\mu} \in \Lambda_k^+$  (which as before we shall identify with the set of conjugacy classes of homomorphisms  $\Gamma_k \rightarrow G$ ) gives rise to a homomorphism  $\Gamma_k \rightarrow G \times \text{GL}(2)$  and thus to an action of  $\Gamma_k$  on  $\text{Bun}_G(\mathbb{A}^2)$  (strictly speaking for this we have to choose a lifting of  $\bar{\mu}$  to  $\text{Hom}(\Gamma_k, G)$ ; however, for different choices of this lifting (lying automatically in the same conjugacy class) we obtain isomorphic actions of  $\Gamma_k$ ). We denote the fixed point set of this action by  $\text{Bun}_{G, \bar{\mu}}(\mathbb{A}^2/\Gamma_k)$ .

It follows from the fact the elements of  $\text{Bun}_G(\mathbb{A}^2)$  have no non-trivial automorphisms that every  $\mathcal{F} \in \text{Bun}_{G, \bar{\mu}}(\mathbb{A}^2/\Gamma_k)$  is actually  $\Gamma_k$ -equivariant bundle on  $\mathbf{S} = \mathbb{P}^2$ . In particular, for every  $\mathcal{F} \in \text{Bun}_{G, \bar{\mu}}(\mathbb{A}^2/\Gamma_k)$  the group  $\Gamma_k$  acts on the fiber of  $\mathcal{F}_0$  of  $\mathcal{F}$  at the  $\Gamma_k$ -fixed point  $0 \in \mathbb{A}^2$ . Hence  $\mathcal{F}_0$  defines a conjugacy class of maps  $\Gamma_k \rightarrow G$ , i.e. an element of  $\Lambda_k^+$ . For every  $\bar{\lambda} \in \Lambda_k^+$  we denote by  $\text{Bun}_{G, \bar{\mu}}^{\bar{\lambda}}(\mathbb{A}^2/\Gamma_k)$  the subset of  $\text{Bun}_{G, \bar{\mu}}(\mathbb{A}^2/\Gamma_k)$  formed by all  $\mathcal{F} \in \text{Bun}_{G, \bar{\mu}}(\mathbb{A}^2/\Gamma_k)$  such that  $\mathcal{F}_0$  is of class  $\bar{\lambda}$ . Clearly, it is a union of connected components of the fixed point set  $\text{Bun}_{G, \bar{\mu}}(\mathbb{A}^2/\Gamma_k)$ . We denote by  $\text{Bun}_{G, \bar{\mu}}^{\bar{\lambda}, a}(\mathbb{A}^2/\Gamma_k)$  the intersection of  $\text{Bun}_{G, \bar{\mu}}^{\bar{\lambda}}(\mathbb{A}^2/\Gamma_k)$  with  $\text{Bun}_G^a(\mathbb{A}^2)$ .

**Conjecture 4.5.**  $\text{Bun}_{G, \bar{\mu}}^{\bar{\lambda}, a}(\mathbb{A}^2/\Gamma_k)$  is connected (possibly empty).

**4.6. Remark.** Most of the results below rest upon this conjecture, so we assume it in what follows. In fact, later we are going to see that Conjecture 4.5 holds in the following cases:

- 1) In the case  $k = 1$  (by [4]).
- 2) In the case  $G = \text{SL}(N)$ . In this case Conjecture 4.5 follows from the well-known results of W. Crawley-Boevey on connectedness of some quiver varieties [9], cf. Section 7 for the details.
- 3) For  $k \gg 0$  (i.e. if we fix  $a$ ,  $\bar{\lambda}$  and  $\bar{\mu}$  and make  $k$  sufficiently large).

4) In the case when  $\bar{\lambda} \geq \bar{\mu}$  we can't quite prove Conjecture 4.5 in full generality but we shall show in Section 5 that in this case we can single out some specific connected component of  $\text{Bun}_{G,\bar{\mu}}^{\bar{\lambda},a}(\mathbb{A}^2)$  which is, in fact, sufficient for our purposes.

**Definition 4.7.** (1) We define  $\mathcal{U}_{G,\bar{\mu}}^{\bar{\lambda},a}(\mathbb{A}^2/\Gamma_k)$  as the closure of  $\text{Bun}_{G,\bar{\mu}}^{\bar{\lambda},a}(\mathbb{A}^2/\Gamma_k)$  inside  $\mathcal{U}_G^a(\mathbb{A}^2)$ .

(2) Let  $\mu = (k, \bar{\mu}, m)$ ,  $\lambda = (k, \bar{\lambda}, l)$  be two elements of  $\Lambda_{\text{aff}}^+$ . Then we set

$$\text{Bun}_{G,\mu}^{\lambda}(\mathbb{A}^2/\Gamma_k) = \text{Bun}_{G,\bar{\mu}}^{\bar{\lambda},a}(\mathbb{A}^2/\Gamma_k) \text{ where } a = k(l - m) + \frac{(\bar{\lambda}, \bar{\lambda})}{2} - \frac{(\bar{\mu}, \bar{\mu})}{2}. \quad (4.3)$$

Similarly, we define  $\mathcal{U}_{G,\mu}^{\lambda}(\mathbb{A}^2/\Gamma_k)$  as the closure  $\text{Bun}_{G,\mu}^{\lambda}(\mathbb{A}^2/\Gamma_k)$  inside  $\mathcal{U}_G^a(\mathbb{A}^2)$ .

The choice of  $a$  in (4.3) might seem a bit bizarre at the first glance; it can be partly justified by the following result:

**Theorem 4.8.** The dimension of  $\text{Bun}_{G,\mu}^{\lambda}(\mathbb{A}^2/\Gamma_k)$  (and thus of  $\mathcal{U}_{G,\mu}^{\lambda}(\mathbb{A}^2/\Gamma_k)$ ) is equal to  $2|\lambda - \mu|$ . In particular,  $\text{Bun}_{G,\mu}^{\lambda}$  is empty unless  $\lambda \geq \mu$ .

Theorem 4.8 is proved in Section 7.8. Its proof is in fact unconditional (i.e. it does not assume the validity of Conjecture 4.5), so by “dimension” we actually mean “the dimension of every connected component of  $\text{Bun}_{G,\mu}^{\lambda}(\mathbb{A}^2/\Gamma_k)$ ”.

Another motivation for (4.3) is given by Theorem 5.2.

**4.9. Factorization and extremal connected components.** Here we are going to give another (perhaps more intuitive) definition of  $\text{Bun}_{G,\mu}^{\lambda}(\mathbb{A}^2/\Gamma_k)$ . The factorization morphism  $\pi^a : \text{Bun}_G^a(\mathbb{A}^2) \rightarrow \mathbb{A}^{(a)}$  is  $\Gamma_k$ -equivariant (with respect to the action  $z \mapsto \zeta z$  of  $\Gamma_k$  on  $\mathbb{A}^1$ ), and gives rise to the factorization morphism

$$\pi_{\bar{\mu}}^{\bar{\lambda},a} : \text{Bun}_{G,\bar{\mu}}^{\bar{\lambda},a}(\mathbb{A}^2/\Gamma_k) \rightarrow (\mathbb{A}^{(a)})^{\Gamma_k} \simeq \bar{\mathbb{A}}^{(b)},$$

where  $b = [a/k]$  is the integral part of  $a/k$ , and  $\bar{\mathbb{A}}^1 = \mathbb{A}^1/\Gamma_k$  is also isomorphic to the affine line. However, there is no a priori reason for  $\pi_{\bar{\mu}}^{\bar{\lambda},a}$  to be surjective. In fact, assuming Conjecture 4.5 now we have the following:

**Lemma 4.10.** Let  $\bar{\lambda}$  and  $\bar{\mu}$  and  $a$  be such that  $\text{Bun}_{G,\bar{\mu}}^{\bar{\lambda},a}$  is non-empty. Then

- (1) There exists unique  $d \in \mathbb{N}$  such that  $\frac{a-d}{k} \in \mathbb{N}$  and such that the image of  $\pi_{\bar{\mu}}^{\bar{\lambda},d}$  consists of one point.
- (2) In the above case let us set  $c = \frac{a-d}{k}$ . Then the image of  $\pi_{\bar{\mu}}^{\bar{\lambda},a}$  is isomorphic to  $\bar{\mathbb{A}}^{(c)}$  which is embedded into  $\bar{\mathbb{A}}^{(b)}$  by adding a multiple of  $0 \in \bar{\mathbb{A}}^1$ .

*Proof.* Choose some  $c$  such that  $a - kc \geq 0$  and embed  $\bar{\mathbb{A}}^{(c)}$  into  $\bar{\mathbb{A}}^{(b)}$  in the way described above. Assume now that there exists a divisor  $D \in \bar{\mathbb{A}}^{(c)}$  disjoint from  $0 \in \bar{\mathbb{A}}^1$  which lies in the image of  $\pi_{\bar{\mu}}^{\bar{\lambda},a}$ .

Then

$$(\pi_{\bar{\mu}}^{\bar{\lambda},a})^{-1}(D) \simeq (\pi_{\bar{\mu}}^{\bar{\lambda},a-kc})^{-1}(0) \times (\pi^c)^{-1}(D) \quad (4.4)$$

(here  $\pi^c : \text{Bun}_G^c(\mathbb{A}^2) \rightarrow \mathbb{A}^{(c)}$  is the usual factorization morphism, and we somewhat abusively identify  $\mathbb{A}^{(c)}$  and  $\overline{\mathbb{A}}^{(c)}$ ). Hence if we choose  $c$  to be the maximal number with this property (i.e. that  $D$  as above exists) and if we set  $d = a - kc$ , then the image of the factorization morphism  $\pi_{\overline{\mu}}^{\overline{\lambda}, d}$  is just one point 0. The uniqueness of  $d$  as well as the second assertion of Lemma 4.10 follows immediately from the factorization isomorphism (4.4).  $\square$

Loosely speaking, using the factorization isomorphism (4.4) the variety  $\text{Bun}_{G, \overline{\mu}}^{\overline{\lambda}, a}(\mathbb{A}^2/\Gamma_k)$  can be generically expressed in terms of  $\text{Bun}_{G, \overline{\mu}}^{\overline{\lambda}, d}(\mathbb{A}^2/\Gamma_k)$ , and of the usual  $\text{Bun}_G(\mathbb{A}^2)$ .

Thus for each couple  $\overline{\mu}, \overline{\lambda} \in \Lambda_k^+$  there exist certain *extremal components*  $\text{Bun}_{G, \overline{\mu}}^{\overline{\lambda}, d}(\mathbb{A}^2/\Gamma_k)$  such that the image of factorization morphism is just one point, and any other component with the same  $\overline{\mu}$  and  $\overline{\lambda}$  is of the kind  $\text{Bun}_{G, \overline{\mu}}^{\overline{\lambda}, d+kc}(\mathbb{A}^2/\Gamma_k)$  for some  $c \in \mathbb{N}$ .

**Conjecture 4.11.** (a) For any  $\overline{\mu}, \overline{\lambda} \in \Lambda_k^+$  there is a unique extremal component  $\text{Bun}_{G, \overline{\mu}}^{\overline{\lambda}, d}(\mathbb{A}^2/\Gamma_k)$  to be denoted  $\min \text{Bun}_{G, \overline{\mu}}^{\overline{\lambda}}(\mathbb{A}^2/\Gamma_k)$ .

(b) There is a lift  $\overline{\mu}_\infty : \mathbb{G}_m \rightarrow G$  (resp.  $\overline{\lambda}_\infty : \mathbb{G}_m \rightarrow G$ ) of  $\overline{\mu} : \Gamma_k \rightarrow G$  (resp.  $\overline{\lambda} : \Gamma_k \rightarrow G$ ) such that  $\min \text{Bun}_{G, \overline{\mu}}^{\overline{\lambda}}(\mathbb{A}^2/\Gamma_k) = \text{Bun}_{G, \overline{\mu}_\infty}^{\overline{\lambda}_\infty}(\mathbb{A}^2/\mathbb{G}_m)$ .<sup>6</sup>

**4.12. Remark.** We will see in Section 5 that  $\text{Bun}_{\overline{\mu}_\infty}^{\overline{\lambda}_\infty}(\mathbb{A}^2/\mathbb{G}_m)$  is isomorphic to  $\mathcal{W}_{\overline{\mu}_\infty}^{\overline{\lambda}_\infty}$ .

In the rest of this Section we shall assume Conjecture 4.11 as well. Note that as before Conjecture 4.11 is obvious for  $k=1$  (by [4]) and we shall see that it also holds true for  $G = \text{SL}(n)$  in Section 7.

**4.13. The main conjecture.** Let  $\overline{\lambda}, \overline{\mu} \in \Lambda_k^+$  and choose a lift  $\lambda = (k, \overline{\lambda}, l)$  of  $\overline{\lambda}$  to a dominant weight of  $G_{\text{aff}}^\vee$ . Note that any other choice  $\lambda'$  equals  $\lambda + n\delta$  where  $\delta = (0, 0, 1)$  is the minimal imaginary root of  $G_{\text{aff}}^\vee$ , and  $n \in \mathbb{Z}$ . Among all the possible lifts  $\mu$  of  $\overline{\mu}$  there is a unique *maximal* one  $\mu_0$  such that  $\mu_0$  has a nonzero multiplicity in the integrable module  $L(\lambda)$  with highest weight  $\lambda$ , while  $\mu_0 + \delta$  has zero multiplicity. The set of lifts  $\mu_0 - \mathbb{N}\delta$  is called a *string* in the terminology of integrable modules. To identify it with a corresponding factorization string of connected components of  $\text{Bun}_{G, \overline{\mu}}^{\overline{\lambda}}(\mathbb{A}^2/\Gamma_k)$ , we will denote  $\text{Bun}_{G, \overline{\mu}}^{\overline{\lambda}, d+kc}(\mathbb{A}^2/\Gamma_k)$ <sup>7</sup> by  $'\text{Bun}_{G, \mu_0 - c\delta}^\lambda(\mathbb{A}^2/\Gamma_k)$  provisionally. So for  $\mu$  in the  $\overline{\mu}$ -string of  $L(\lambda)$  we shall denote by  $'\text{Bun}_{G, \mu}^\lambda(\mathbb{A}^2/\Gamma_k)$  the corresponding connected component. We shall also denote by  $'\mathcal{U}_{G, \mu}^\lambda(\mathbb{A}^2/\Gamma_k)$  its closure in the Uhlenbeck space of  $\mathbb{A}^2$ . Note that  $'\mathcal{U}_{G, \mu+l\delta}^{\lambda+l\delta}(\mathbb{A}^2/\Gamma_k) = '\mathcal{U}_{G, \mu}^\lambda(\mathbb{A}^2/\Gamma_k)$  for arbitrary  $l \in \mathbb{Z}$ .

The main claim of this paper is this:

- One can (and should) think about the varieties  $\mathcal{U}_{G, \mu}^\lambda(\mathbb{A}^2/\Gamma_k)$  as analogs of the transversal slices  $\overline{\mathcal{W}}_{G, \mu}^\lambda$  we considered in Section 2.8; similarly one should think about  $\text{Bun}_{G, \mu}^\lambda$ 's as affine analogs of the varieties  $\mathcal{W}_{G, \mu}^\lambda$ .

<sup>6</sup>The definition of  $\text{Bun}_G(\mathbb{A}^2/\mathbb{G}_m)$  is essentially the same as that of  $\text{Bun}_G(\mathbb{A}^2/\Gamma_k)$ ; more details can be found in Section 5.

<sup>7</sup>Here, as before, we have  $\text{Bun}_{G, \overline{\mu}}^{\overline{\lambda}, d}(\mathbb{A}^2/\Gamma_k) = \min \text{Bun}_{G, \overline{\mu}}^{\overline{\lambda}}(\mathbb{A}^2/\Gamma_k)$ .

Below is the main conjecture of this paper, which partially justifies the above principle. Some other (perhaps more convincing) evidence may also be found in Section 5.

**Conjecture 4.14.** (1) For  $\mu = (k, \bar{\mu}, m)$ ,  $\lambda = (k, \bar{\lambda}, l)$  we have  ${}^t\text{Bun}_{G,\mu}^\lambda(\mathbb{A}^2/\Gamma_k) = \text{Bun}_{G,\mu}^\lambda(\mathbb{A}^2/\Gamma_k) := \text{Bun}_{G,\bar{\mu}}^{\bar{\lambda},a}(\mathbb{A}^2/\Gamma_k)$  where  $a = k(l - m) + \frac{(\bar{\lambda}, \bar{\lambda})}{2} - \frac{(\bar{\mu}, \bar{\mu})}{2}$ . Hence  $\mathcal{U}_{G,\mu}^\lambda(\mathbb{A}^2/\Gamma_k) = \mathcal{U}_{G,\mu}^\lambda(\mathbb{A}^2/\Gamma_k)$ .

(2) We have  $\mathcal{U}_{G,\mu}^\nu(\mathbb{A}^2/\Gamma_k) \subset \mathcal{U}_{G,\mu}^\lambda(\mathbb{A}^2/\Gamma_k)$  if and only if  $\lambda \geq \nu$  (that is,  $\lambda - \nu$  is a positive linear combination of simple roots of  $\mathfrak{g}_{\text{aff}}^\vee$ ). The corresponding closed embedding will be called “adding defect  $(\lambda - \nu) \cdot (0, 0)$ ” (note that this part of the conjecture is compatible with Theorem 4.8).

(3) Let  $V_\mu^\lambda$  denote the stalk of  $\text{IC}(\mathcal{U}_{G,\mu}^\lambda(\mathbb{A}^2/\Gamma_k))$  at the unique torus-fixed point  $pt$ . This is a graded vector space. Then  $V_\mu^\lambda$  is concentrated in even degrees and we have a canonical isomorphism

$$(V_\mu^\lambda)_{-2i} \simeq \text{gr}_i^F L(\lambda)_\mu$$

(see Section 3.4 for the definition of filtration  $F$ ).

Note that  $\mathcal{U}_{G,\mu}^\lambda$  admits a  $\mathbb{C}^*$ -action which contracts it to  $pt$ . Thus the global intersection cohomology  $\text{IH}^*(\mathcal{U}_{G,\mu}^\lambda)$  is also equal to  $V_\mu^\lambda$ .

**4.15. The  $q$ -analog of the Kostant multiplicity formula in the affine case.** We now want to describe the affine analog of Section 2.6. In the same way as in Section 2.6 let us introduce the generating functions:

$$\text{IC}_\mu^\lambda(q) := \sum_i \dim(V_\mu^\lambda)_{-2i} q^i; \quad {}^e C_\mu^\lambda(q) := \sum_i \dim \text{gr}_i^F L(\lambda)_\mu q^i.$$

Then the last assertion of Conjecture 4.14 says that  $\text{IC}_\mu^\lambda(q) = {}^e C_\mu^\lambda$ . We shall refer to this function as *the  $q$ -character* of  $L(\lambda)_\mu$ .

The  $q$ -character of  $L(\lambda)_\mu$  can be given another purely combinatorial definition. Namely, let  $\Lambda_{\text{aff}}^{\text{pos}}$  denote the subsemigroup of  $\Lambda_{\text{aff}}$  consisting of elements of the form  $\sum a_j \alpha_j$  where  $a_j \in \mathbb{Z}_{\geq 0}$  and  $\alpha_j$  are positive coroots of  $G_{\text{aff}}$  (which is the same as positive roots of  $G_{\text{aff}}^\vee$ ). Note that  $\Lambda_{\text{aff}}^{\text{pos}} \subset \Lambda_{\text{aff},0}$ . Let also  $R_{\text{aff}}$  denote the set of all roots of  $G_{\text{aff}}^\vee$  and let  $R_{\text{aff}}^+$  be the corresponding set of positive roots. For all  $\beta \in \Lambda^{\text{pos}}$  let us introduce the Kostant partition function of  $K_\beta(q)$  of  $\beta$  by:

$$\sum_{\beta \in \Lambda_{\text{aff}}^{\text{pos}}} K_\beta(q) := \prod_{\alpha \in R_{\text{aff}}^+} (1 - qe^\alpha)^{-1},$$

Now we define following [26]

$$C_\mu^\lambda(q) := \sum_{w \in W_{\text{aff}}} (-1)^{\ell(w)} K_{w \cdot \lambda - \mu}(q). \quad (4.5)$$

Then we have the following

**Conjecture 4.16.** For every  $\lambda, \mu \in \Lambda_{\text{aff}}^+$  We have  $\text{IC}_\mu^\lambda(q) = {}^e C_\mu^\lambda(q) = C_\mu^\lambda(q)$ .



Note that the equality  ${}^e C_\mu^\lambda(q) = C_\mu^\lambda(q)$  is purely representation-theoretic, i.e. it doesn't involve any geometry in it. This equality is the natural affine generalization of the main result of [8]; in some special cases it was proved in [10] (in fact, I. Grojnowski told us that the methods of [10] allow to prove the equality  ${}^e C_\mu^\lambda(q) = C_\mu^\lambda(q)$  in general but we were not able to reconstruct that proof). It is also natural to expect that with the appropriate modifications (as in Theorem 2.9 in [15]) the equality  ${}^e C_\mu^\lambda(q) = C_\mu^\lambda(q)$  can be generalized to the case when  $\mu$  is not necessarily dominant.

**4.17. The full slices.** In Section 2.8 we introduced also the infinite-dimensional varieties  $\mathcal{W}_\mu$  (in that case  $\mu$  was an element of  $\Lambda$ , not of  $\Lambda_{\text{aff}}$ ); more precisely,  $\mathcal{W}_\mu$  was an ind-scheme and it might be thought of as a transversal slice to  $\text{Gr}_G^\mu$  in  $\text{Gr}_G$ . It turns out that we may construct an analog of  $\mathcal{W}_{G,\mu}$  in the affine case.

Namely, we consider the inductive limit of the system of embeddings  $\mathcal{U}_{G,\mu}^\lambda(\mathbb{A}^2/\Gamma_k) \hookrightarrow \mathcal{U}_{G,\mu}^{\lambda+c\delta}(\mathbb{A}^2/\Gamma_k)$  (for  $c \in \mathbb{N}$ ; the embedding is just twisting by a multiple of  $(0,0) \in \overline{\Lambda}^2$ ), and denote it by  $\mathcal{U}_{G,\mu}(\mathbb{A}^2/\Gamma_k)$ . Note that the limit does not depend on  $\lambda$ , as suggested by the notation (for various choices of  $\lambda$  we obtain cofinal systems).

## 5. THE CASE OF $k \gg 0$

**5.1.** We want to understand what happens when we fix  $\bar{\lambda}, \bar{\mu}$  and  $a$  and make  $k$  very large. Let us choose some  $\bar{\mu} : \mathbb{G}_m \rightarrow T \subset G$ . As before it defines an action of  $\mathbb{G}_m$  on  $\text{Bun}_G^a(\mathbb{A}^2)$ . Thus for each  $\bar{\lambda}, \bar{\mu} \in \Lambda^+$  and  $a \geq 0$  we may speak of  $\text{Bun}_{G,\bar{\mu}}^{\bar{\lambda},a}(\mathbb{A}^2/\mathbb{G}_m)$  and we let  $\mathcal{U}_{G,\bar{\mu}}^{\bar{\lambda},a}(\mathbb{A}^2/\mathbb{G}_m)$  denote its closure in  $\mathcal{U}_G^a(\mathbb{A}^2)$ . It is obvious that the factorization mapping  $\pi_{\bar{\mu}}^{\bar{\lambda},a}$  sends all of  $\mathcal{U}_{G,\bar{\mu}}^{\bar{\lambda},a}(\mathbb{A}^2/\mathbb{G}_m)$  to the divisor  $a \cdot 0$ .

On the other hand, it is clear that for  $k \gg 0$  we have  $(\text{Bun}_G^a(\mathbb{A}^2))^{\Gamma_k} = (\text{Bun}_G^a)^{\mathbb{G}_m}$ . Hence if  $k$  is large we have  $\text{Bun}_{G,\bar{\mu}}^{\bar{\lambda},a}(\mathbb{A}^2/\Gamma_k) = \text{Bun}_{G,\bar{\mu}}^{\bar{\lambda},a}(\mathbb{A}^2/\mathbb{G}_m)$ .

**Theorem 5.2.** (1) *The variety  $\text{Bun}_{G,\bar{\mu}}^{\bar{\lambda},a}(\mathbb{A}^2/\mathbb{G}_m)$  is empty unless  $\lambda \geq \mu$  and  $a = \frac{(\bar{\lambda}, \bar{\lambda})}{2} - \frac{(\bar{\mu}, \bar{\mu})}{2}$ .*

(2) *In the above case (i.e. when  $a = \frac{(\bar{\lambda}, \bar{\lambda})}{2} - \frac{(\bar{\mu}, \bar{\mu})}{2}$ ) there are canonical isomorphisms*

$$\text{Bun}_{G,\bar{\mu}}^{\bar{\lambda},a}(\mathbb{A}^2/\mathbb{G}_m) \simeq \mathcal{W}_{G,\bar{\mu}}^{\bar{\lambda}}; \quad \mathcal{U}_{G,\bar{\mu}}^{\bar{\lambda},a}(\mathbb{A}^2/\mathbb{G}_m) \simeq \overline{\mathcal{W}}_{G,\bar{\mu}}^{\bar{\lambda}}.$$

(3) *Assume that  $\bar{\lambda} \geq \bar{\mu}$  and  $\bar{\lambda}, \bar{\mu} \in \Lambda_k^+$  for some  $k > 0$ . Let  $a = \frac{(\bar{\lambda}, \bar{\lambda})}{2} - \frac{(\bar{\mu}, \bar{\mu})}{2}$ . Assume that  $k > a$ . Then  $\text{Bun}_{G,\bar{\mu}}^{\bar{\lambda},b}(\mathbb{A}^2/\mathbb{G}_m)$  is empty if  $b < a$ .*

*Proof.* Let us first concentrate on the first two assertions of Theorem 5.2. Let  $\text{Gr}_{G,BD}^n$  denote the Beilinson-Drinfeld Grassmannian classifying the following data:

- a) An effective divisor  $D = \sum a_i x_i$  on  $\mathbb{A}^1$  of degree  $n$ .
- b) A  $G$ -bundle  $\mathcal{F}$  on  $\mathbb{P}^1$  trivialized away from the support of  $D$ .

It is well-known that  $\text{Gr}_{G,BD}^n$  has a natural structure of an ind-scheme. Note that we have a natural embedding  $\text{Gr}_G^n \hookrightarrow \text{Gr}_{G,BD}^n$ ; the image of this embedding consists of all the points of  $\text{Gr}_{G,BD}^n$  for which all  $x_i$  are equal to zero.

Let  $\mathbf{C}$  denote another copy of  $\mathbb{P}^1$ ; we denote by  $\infty_{\mathbf{C}}$  and  $0_{\mathbf{C}}$  the corresponding 0 and  $\infty$  points. Recall from [4] that  $\text{Bun}_G^a(\mathbb{A}^2)$  can be identified with the space  $\text{Maps}^a(\mathbf{C}, \text{Gr}_{G,BD}^a)^0$  of all maps from  $\mathbf{C} \simeq \mathbb{P}^1$  to  $\text{Gr}_{G,BD}^a$  which

- 1) Have degree  $a$  in the appropriate sense;
- 2) Send  $\infty_{\mathbf{C}}$  to a point of  $\text{Gr}_{G,BD}^a$  corresponding to  $\mathcal{F}$  being trivial bundle with the standard trivialization (with  $D$  being arbitrary).

Assume now that we are given a point of  $\text{Bun}_{G,\bar{\mu}}^{\bar{\lambda},a}(\mathbb{A}^2/\mathbb{G}_m)$ . Then it is clear that the map  $\pi^a$  sends this point to the divisor  $a \cdot 0$ ; also, the divisor  $D$  must necessarily be of the form  $a \cdot 0_{\mathbf{C}}$ , that means that this points corresponds to a mapping  $f : \mathbf{C} \rightarrow \text{Gr}_G$ . The map  $f$  must satisfy the following properties:

- $f$  is of degree  $a$
- $f(\infty_{\mathbf{C}})$  is the unique  $G(\mathcal{O})$ -invariant point of  $\text{Gr}_G$ .
- $f(\tau z) = \bar{\mu}(\tau^{-1})(f(z))^{\tau^{-1}}$  where the super-script  $\tau^{-1}$  in the right hand side stands for the action of  $\tau^{-1}$  by loop rotation on  $\text{Gr}_G$ .
- $s^{\bar{\mu}} f(0_{\mathbf{C}})$  lies in  $(\text{Gr}_G^{\bar{\lambda}})^{\mathbb{C}^*}$  (this follows from Lemma 2.3).

It is clear that such an  $f$  is uniquely determined by its value at  $1 \in \mathbf{C}$ . Let us consider the map  $g = s^{\bar{\mu}} \cdot f$ . Then  $g$  is again a map of degree  $a$  which satisfies

- a)  $g(\tau z) = (g(z))^{\tau^{-1}}$ ,
- b)  $g(\infty_{\mathbf{C}}) = s^{\bar{\mu}}$ ,
- c)  $g(0_{\mathbf{C}}) \in (\text{Gr}_G^{\bar{\lambda}})^{\mathbb{C}^*}$ .

The map  $g$  again is uniquely determined by its value at 1. In other words, evaluation of  $g$  at 1 gives rise to an embedding  $\eta$  of  $\text{Bun}_{G,\bar{\mu}}^{\bar{\lambda},a}(\mathbb{A}^2/\mathbb{G}_m)$  into  $\text{Gr}_G$ .

**Lemma 5.3.** (1) *The space of maps  $g$  as above is empty unless  $a = \frac{(\bar{\lambda}, \bar{\lambda})}{2} - \frac{(\bar{\mu}, \bar{\mu})}{2}$ .*  
(2) *In the latter case the embedding  $\eta$  is an isomorphism between  $\text{Bun}_{G,\bar{\mu}}^{\bar{\lambda},a}(\mathbb{A}^2/\Gamma_k)$  and  $\mathcal{W}_{G,\bar{\mu}}^{\bar{\lambda}}$ .*

*Proof.* It is well-known (cf. [11]) that  $\text{Pic}(\text{Gr}_G) \simeq \mathbb{Z}$ ; different line bundles on  $\text{Gr}_G$  correspond canonically to different integral even symmetric invariant bilinear forms on  $\mathfrak{g}$ . Let  $\mathcal{L}$  denote the generator of  $\text{Pic}(\text{Gr}_G)$  corresponding to our form  $(\cdot, \cdot)$ . Then  $\deg(g)$  is by definition the degree of the line bundle  $g^*\mathcal{L}$ . We claim that for any map  $g$  satisfying the properties a), b), c) above one has  $\deg(g^*\mathcal{L}) = \frac{(\bar{\lambda}, \bar{\lambda})}{2} - \frac{(\bar{\mu}, \bar{\mu})}{2}$ . Indeed, it follows from a) that the line bundle  $g^*\mathcal{L}$  on  $\mathbf{C} \simeq \mathbb{P}^1$  is  $\mathbb{G}_m$ -equivariant. It also follows from b) and c) that the action of  $\mathbb{G}_m$  in the fiber of  $g^*\mathcal{L}$  at  $\infty_{\mathbf{C}}$  is given by  $\tau \mapsto \tau^{\frac{(\bar{\lambda}, \bar{\lambda})}{2}}$  and the action of  $\mathbb{G}_m$  in the fiber of  $g^*\mathcal{L}$  at  $0_{\mathbf{C}}$  is given by  $\tau \mapsto \tau^{\frac{(\bar{\mu}, \bar{\mu})}{2}}$ . It is easy to see (according to the standard classification of  $\mathbb{G}_m$ -equivariant line bundles on  $\mathbb{P}^1$ ) that this implies that  $\deg(g^*\mathcal{L}) = \frac{(\bar{\lambda}, \bar{\lambda})}{2} - \frac{(\bar{\mu}, \bar{\mu})}{2}$ . This proves the first assertion of Lemma 5.3.

To prove the second assertion, let us note that a point  $x$  in  $\text{Gr}_G$  lies in  $\mathcal{W}_{G,\bar{\mu}}^{\bar{\lambda}}$  if and only the following two properties are satisfied:

$$\lim_{\tau \rightarrow 0} x^\tau \in \text{Gr}_G^{\bar{\lambda}}. \quad (5.1)$$

and

$$\lim_{\tau \rightarrow \infty} x^\tau = s^{\bar{\mu}}. \quad (5.2)$$

Let us now prove that  $\eta$  lands in  $\mathcal{W}_{G,\bar{\mu}}^{\bar{\lambda}}$ . Indeed, let  $g$  be as above and set  $x = g(1)$ . Then a), b) and c) above imply that  $x$  satisfies (5.1) and (5.2) which proves what we need.

On the other hand let us show that the map  $\eta : \text{Bun}_{G,\bar{\mu}}^{\bar{\lambda},a} \rightarrow \mathcal{W}_{G,\bar{\mu}}^{\bar{\lambda}}$  is surjective. Take any  $x \in \mathcal{W}_{G,\bar{\mu}}^{\bar{\lambda}}$  and set  $g(z) = x^{z^{-1}}$  for any  $z \in \mathbb{C}^*$ . Since  $\text{Gr}_G$  is ind-proper, it follows that  $g$  extends to a map  $\mathbb{P}^1 \rightarrow \text{Gr}_G$  which automatically satisfies a). Now the equations (5.1) and (5.2) imply that  $g$  also satisfies b) and c).  $\square$

The first assertion of Lemma 5.3 proves Theorem 5.2(1). To prove the second assertion of Theorem 5.2 we must show that  $\eta$  extends to an isomorphism between  $\mathcal{U}_{G,\bar{\mu}}^{\bar{\lambda},a}(\mathbb{A}^2/\mathbb{G}_m)$  and  $\overline{\mathcal{W}}_{G,\bar{\mu}}^{\bar{\lambda}}$ . Actually, the definition of  $\eta$  extends word-by-word to  $\mathcal{U}_{G,\bar{\mu}}^{\bar{\lambda},a}$ ; it is also easy to show in the same way as before that in this way we get a map  $\bar{\eta} : \mathcal{U}_{G,\bar{\mu}}^{\bar{\lambda},a} \rightarrow \overline{\mathcal{W}}_{G,\bar{\mu}}^{\bar{\lambda}}$ . We now must prove that this map is an isomorphism.

However, the stratification (4.1) of  $\mathcal{U}_G^a(\mathbb{A}^2)$  implies that

$$\mathcal{U}_{G,\bar{\mu}}^{\bar{\lambda},a}(\mathbb{A}^2/\mathbb{G}_m) \simeq \bigsqcup_{\bar{\mu} \leq \bar{\lambda}' \leq \bar{\lambda}} \text{Bun}_{G,\bar{\mu}}^{\bar{\lambda}', \frac{(\bar{\lambda}', \bar{\lambda}')}{2} - \frac{(\bar{\mu}, \bar{\mu})}{2}}(\mathbb{A}^2/\mathbb{G}_m).$$

Similarly,

$$\overline{\mathcal{W}}_{G,\bar{\mu}}^{\bar{\lambda}} = \bigsqcup_{\bar{\mu} \leq \bar{\lambda}' \leq \bar{\lambda}} \mathcal{W}_{G,\bar{\mu}}^{\bar{\lambda}'}.$$

This implies that  $\bar{\eta}$  is a bijection on closed points. Note that the union of the boundary strata in  $\overline{\mathcal{W}}_{G,\bar{\mu}}^{\bar{\lambda}}$  is of codimension at least 2; hence  $\bar{\eta}$  is an isomorphism away from codimension 2.

On the other hand,  $\overline{\mathcal{W}}_{G,\bar{\mu}}^{\bar{\lambda}}$  is known to be normal (cf. [11] and references therein). This implies that  $\bar{\eta}^{-1}$  is defined everywhere and since  $\bar{\eta}$  is a bijection on closed points it follows that  $\bar{\eta}$  is an isomorphism.

The proof of the third assertion of Theorem 5.2 is a word-by-word repetition of the proof of the first assertion of Lemma 5.3.  $\square$

**5.4. The case of large  $k$ .** Assume that we have  $\bar{\lambda} \geq \bar{\mu}$  such that  $\bar{\lambda}, \bar{\mu} \in \Lambda_k^+$ . Then Theorem 5.2 produces certain connected component of  $\text{Bun}_{G,\bar{\mu}}^{\bar{\lambda},a}(\mathbb{A}^2/\Gamma_k)$  for  $a = \frac{(\bar{\lambda}, \bar{\lambda})}{2} - \frac{(\bar{\mu}, \bar{\mu})}{2}$  — namely, the one which contains  $\text{Bun}_{G,\bar{\mu}}^{\bar{\lambda},a}$ . Moreover, it follows from Theorem 5.2 that for  $\frac{(\bar{\lambda}, \bar{\lambda})}{2} - \frac{(\bar{\mu}, \bar{\mu})}{2} < k$  this component is extremal. We don't know how to prove that there are no other extremal components in this case; however, it follows from Theorem 5.2 that in this case we have  $\min \text{Bun}_{G,\bar{\mu}}^{\bar{\lambda}}(\mathbb{A}^2/\Gamma_k) = \text{Bun}_{G,\bar{\mu}}^{\bar{\lambda},a}(\mathbb{A}^2/\Gamma_k)$ . This proves Conjecture 4.14(1) under the above assumptions. Indeed, let us choose as before a lift  $\lambda = (k, \bar{\lambda}, m)$  of  $\bar{\lambda}$  to  $\Lambda_{\text{aff},k}^+$ . Recall that in Section 4.13 we introduced a weight  $\mu_0 \in \Lambda_{\text{aff},k}^+$ , which is a lift of  $\bar{\mu}$  to  $\Lambda_{\text{aff},k}^+$ , and such that  $L(\lambda)_{\mu_0} \neq 0$  and  $L(\lambda)_{\mu_0 + \delta} = 0$ . Then Conjecture 4.14(1) in this case is equivalent to saying that  $\mu = (k, \bar{\mu}, m)$ . To prove this let us note that the integrable

$\mathfrak{g}_{\text{aff}}^\vee$ -module  $L(\lambda)$  contains the finite-dimensional irreducible  $\mathfrak{g}^\vee$ -module  $L(\bar{\lambda})$ . It is equal to the direct sum of all the  $L(\lambda)_\nu$  where  $\nu$  is of the form  $(k, \bar{\nu}, m)$ . Thus, since for a dominant weight  $\bar{\mu}$  satisfying  $\bar{\mu} \leq \bar{\lambda}$  we have  $L(\bar{\lambda})_{\bar{\mu}} \neq 0$ , it follows that  $L(\lambda)_{(k, \bar{\mu}, m)} \neq 0$ . On the other hand, we have  $L(\lambda)_{(k, \bar{\mu}, m) + \delta} = 0$  since  $(k, \bar{\mu}, m) + \delta = (k, \bar{\mu}, m + 1) \not\leq (k, \bar{\lambda}, m) = \lambda$ .

Even though we can't prove the absence of other extremal components in this case, we can now set  $\text{Bun}_{G, \bar{\mu}}^{\bar{\lambda}, a+kc}(\mathbb{A}^2/\Gamma_k)$  to be the *connected subvariety* of  $\text{Bun}_G(\mathbb{A}^2/\Gamma_k)$  obtained by adding defect  $c \cdot (0, 0)$  to the above extremal component and just ignore the other components of  $\text{Bun}_G(\mathbb{A}^2/\Gamma_k)$  if they exist (of course, we believe that they don't exist). In this way, the notation  $\text{Bun}_{G, \mu}^\lambda$  acquires definite meaning, which is compatible with Conjecture 4.14(1).

Also, in this case we can now prove the dimension formula, claimed in the first assertion of Conjecture 4.14(2): it follows immediately from Lemma 2.2(3).

**5.5. Proof of the conjectures for large  $k$ .** Fix as before  $\bar{\lambda}$  and  $\bar{\mu}$  such that  $\bar{\lambda} \geq \bar{\mu}$ . According to Theorem 5.2 the only component of  $\text{Bun}_{G, \bar{\mu}}^{\bar{\lambda}}(\mathbb{A}^2/\Gamma_k)$  which survives in the limit  $k \rightarrow \infty$  is  $\text{Bun}_{G, \bar{\mu}}^{\bar{\lambda}, a}(\mathbb{A}^2/\Gamma_k)$  where  $a = \frac{(\bar{\lambda}, \bar{\lambda})}{2} - \frac{(\bar{\mu}, \bar{\mu})}{2}$ .

It follows that in the limit  $k \rightarrow \infty$  all the conjectures of Section 4 hold in this case, except, possibly Conjecture 4.14 whose proof is explained below.

It is enough for us to assume that  $\bar{\lambda} \geq \bar{\mu}$  such that  $\bar{\lambda}, \bar{\mu} \in \Lambda_k^+$  and  $\lambda = (k, \bar{\lambda}, m), \mu = (k, \bar{\mu}, m)$ . According to the previous subsection, in this case we have

$$L(\lambda)_\mu = L(\bar{\lambda})_{\bar{\mu}}. \quad (5.3)$$

Thus, using the assertion of Theorem 5.2, it follows that in order to prove Conjecture 4.14 in this case, it is enough to check that the filtration on the LHS of (5.3) defined by the principal nilpotent  $e_{\text{aff}}$  of  $\mathfrak{g}_{\text{aff}}^\vee$  coincides with the filtration defined on the RHS of (5.3) by the principal nilpotent  $e$  of  $\mathfrak{g}^\vee$ . This is obvious, since the difference  $e_{\text{aff}} - e$  acts trivially on  $L(\bar{\lambda}) \subset L(\lambda)$ .

## 6. THE CASE OF $k = 1$

In the case  $k = 1$  all the conjectures of Section 4 follow immediately from [4] except for Conjecture 4.14(3). In this Section we explain that Conjecture 4.14(3) can be reduced to an explicit statement in representation theory, which most probably follows from [10], section 11. In particular, in this case we prove that the total stalk  $V_\mu^\lambda$  of  $\text{IC}(\mathcal{U}_{G, \mu}^\lambda(\mathbb{A}^2/\Gamma_k))$  at the canonical point of  $\mathcal{U}_{G, \mu}^\lambda(\mathbb{A}^2/\Gamma_k)$  is isomorphic to *some* graded version of  $L(\lambda)_\mu$  (i.e. we prove the equality  $\text{IC}_\mu^\lambda(1) = {}^e C_\mu^\lambda(1)$ ). We also prove the equality  $\text{IC}_\mu^\lambda(q) = C_\mu^\lambda(q)$  in the *ADE* case.

**6.1. The structure of  $\mathfrak{g}_{\text{aff}}^\vee$ .** Recall that the Langlands dual Lie algebra  $\mathfrak{g}_{\text{aff}}^\vee$  (in general, twisted) admits the following realization. There exist

- 1) A simple Lie algebra  $\mathfrak{g}'$ ,
- 2) Its outer automorphism  $\sigma$  of order  $m = 1, 2, 3$ ,
- 3) A primitive  $m$ -th root of unity  $\omega$

such that

$$\mathfrak{g}_{\text{aff}}^\vee \simeq \mathbb{C}c \oplus \mathbb{C}d \oplus \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}'_n \otimes t^n$$

where  $c$  is the central element,  $d$  is the loop rotations element, and  $\mathfrak{g}'_n := \{x \in \mathfrak{g}' : \sigma(x) = \omega^n x\}$ . Let  $\mathfrak{t}' \subset \mathfrak{g}'$  be a  $\sigma$ -invariant Cartan subalgebra, and let  $\mathfrak{t}'_n := \mathfrak{t}' \cap \mathfrak{g}'_n$ . We set

$$\mathfrak{a}^- := \bigoplus_{n < 0} \mathfrak{t}'_n \otimes t^n.$$

This is naturally a  $\mathbb{Z}^-$ -graded vector space. The symmetric algebra  $\text{Sym } \mathfrak{a}^-$  acquires the induced  $\mathbb{Z}^{\leq 0}$ -grading,  $\text{Sym } \mathfrak{a}^- = \bigoplus_{n \in \mathbb{Z}^{\leq 0}} (\text{Sym } \mathfrak{a}^-)_n$ .

**6.2. The basic integrable module over  $\mathfrak{g}_{\text{aff}}^\vee$ .** Let  $\lambda = (1, 0, 0)$  be the basic fundamental weight of  $\mathfrak{g}_{\text{aff}}^\vee$ . Note that this is the only integrable weight of  $G_{\text{aff}}^\vee$  at level 1 since  $G$  is simply connected. Then the only dominant weights  $\mu$  of  $\mathfrak{g}_{\text{aff}}^\vee$  for which  $L(\lambda)_\mu \neq 0$  are of the form  $\mu = \lambda + n\delta = (1, 0, n)$ ,  $n \in \mathbb{Z}^{\leq 0}$ . According to [19] (it is essential here that  $\mathfrak{g}_{\text{aff}}^\vee$  is Langlands dual of an *untwisted* affine Lie algebra), we have  $L(\lambda)_{\lambda+n\delta} \simeq (\text{Sym } \mathfrak{a}^-)_n$  for any  $n \in \mathbb{Z}^{\leq 0}$ .

Now note that  $\mathfrak{g}'_0$  is another simple Lie algebra. We choose its principal nilpotent element  $e'$ . According to Kostant's theorem,  $\dim(\mathfrak{g}'_m)^{e'} = \dim(\mathfrak{t}'_m)$ . Hence we have a (non-canonical) isomorphism of graded vector spaces  $\mathfrak{a}^- \simeq \bigoplus_{m < 0} (\mathfrak{g}'_m)^{e'}$ . The symmetric algebra  $\text{Sym}(\bigoplus_{m < 0} (\mathfrak{g}'_m)^{e'})$  acquires the induced  $\mathbb{Z}^{\leq 0}$ -grading,  $\text{Sym}(\bigoplus_{m < 0} (\mathfrak{g}'_m)^{e'}) =: \bigoplus_{n \in \mathbb{Z}^{\leq 0}} \text{Sym}_n$ .

**6.3. Comparison with IC-stalks.** According to [4], the total IC stalk  $V_{\lambda+n\delta}^\lambda$  (with grading disregarded) of  $\text{IC}(\mathcal{U}_{G, \lambda+n\delta}^\lambda(\mathbb{A}^2))$  at the canonical point is isomorphic to  $\text{Sym}_n$ . Thus we have proved a noncanonical isomorphism between  $V_{\lambda+n\delta}^\lambda$  and  $L(\lambda)_{\lambda+n\delta}$ , for any  $n \in \mathbb{Z}^{\leq 0}$ .

On the other hand, the vector space  $(\mathfrak{g}'_m)^{e'}$  also has another grading, coming from a Jacobson-Morozov triple  $(e', h', f')$  in  $\mathfrak{g}'_0$  (cf. the end of Section 7 of [4] for more details); this grading extends to each  $\text{Sym}_n$  (making  $\text{Sym}(\bigoplus_{m < 0} (\mathfrak{g}'_m)^{e'})$  a bigraded vector space). According to Theorem 7.10 of [4] the second grading on  $\text{Sym}_n$  (suitably normalized) corresponds to the cohomological grading on  $V_{\lambda+n\delta}^\lambda$ . Comparing with Corollary 2 of [26] we see that in case  $\mathfrak{g}$  is of type *ADE* the generating function of graded dimensions of  $V_{\lambda+n\delta}^\lambda$  coincides with the level one  $t$ -string function  $\alpha_\lambda^\lambda(t)$ , and hence  $\text{IC}_{\lambda+n\delta}^\lambda(q) = C_{\lambda+n\delta}^\lambda(q)$ . Thus, in order to prove the full statement of Conjecture 4.16 in this case we must check that  $\text{Sym}_n$  is isomorphic to  $L(\lambda)_{\lambda+n\delta}$  as *graded* vector space. We don't know how to do this, but according to C. Teleman and I. Grojnowski (private communication) one can probably use the techniques of [10] in order to construct such an isomorphism.

## 7. THE CASE OF $G = \text{SL}(N)$

In this Section we prove most of the conjectures of Section 4 in the case  $G = \text{SL}(N)$ . This Section should be thought of as an affine analog of the paper [23], where the authors discuss the relations between the (nonaffine)  $A_{k-1}$  quiver varieties, the transversal slices in the (classical) affine Grassmannians of  $\text{SL}(N)$ , and Howe's skew  $(\text{GL}(k), \text{GL}(N))$ -duality.

**7.1. Cyclic quiver varieties and their IC-sheaves.** We consider the cyclic quiver with  $k$  vertices corresponding to the characters of  $\Gamma_k$  [24]. For two  $\Gamma_k$ -modules  $V, W$  we denote by  $\mathfrak{M}_0(V, W)$  the affine (singular, in general) Nakajima quiver variety (cf. [24]); <sup>8</sup> we

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<sup>8</sup>Of course here the word “affine” is used in the sense of algebraic geometry and not in the sense of “affine Lie algebras”.

also denote by  $\mathfrak{M}_0^{\text{reg}}(V, W)$  its regular part. Let  $N = \dim W \geq 2$ ,  $a = \dim V$ . We set  $G = \text{SL}(N)$ . Note that in this case  $\mathfrak{g}_{\text{aff}} \simeq \mathfrak{g}_{\text{aff}}^\vee$ .

Let us write  $w = \underline{\dim} W = (w_1, \dots, w_k)$  if

- 1) The multiplicity of the trivial representation of  $\Gamma_k$  in  $W$  is  $w_k$ .
- 2) For  $i = 1, \dots, k-1$  the multiplicity of the character  $\zeta \mapsto \zeta^i$  in  $W$  is  $w_i$ .

Similarly, one may introduce  $v = \underline{\dim} V = (v_1, \dots, v_k)$ . Let  $\omega_i$  be the  $i$ -th fundamental weight of  $\mathfrak{gl}(k)_{\text{aff}}$ ; by definition it is of level 1. In the standard notation we write  $\omega_i = (1, E_1 + \dots + E_i, 0)$ . Let  $\alpha_i$  be the  $i$ -th simple root of  $\mathfrak{gl}(k)_{\text{aff}}$ ; in the standard notation we have  $\alpha_i = (0, E_i - E_{i+1}, 0)$  for  $i = 1, \dots, k-1$ , and  $\alpha_0 = \alpha_k = (0, -E_1 + E_k, 1)$ . Then Nakajima's criterion says that  $\mathfrak{M}_0^{\text{reg}}(V, W)$  is nonempty if and only if the following two properties hold:

- The  $\mathfrak{gl}(k)_{\text{aff}}$ -weight  $\sum_{i=1}^k w_i \omega_i - \sum_{i=1}^k v_i \alpha_i$  is dominant.
- The multiplicity of  $\sum_{i=1}^k w_i \omega_i - \sum_{i=1}^k v_i \alpha_i$  in the integrable  $\mathfrak{gl}(k)_{\text{aff}}$ -module with highest weight  $\sum_{i=1}^k w_i \omega_i$  is non-zero.

In this case Theorem 5.2 of [24] implies the following result:

**Theorem 7.2.** *The stalk of  $\text{IC}(\mathfrak{M}_0(V, W))$  at the most singular point has the following properties:*

- a) *It is concentrated in the even degrees.*
- b) *Its total dimension (with grading disregarded) equals the multiplicity of the integrable  $\mathfrak{gl}(k)_{\text{aff}}$ -module  $L(\sum_{i=1}^k w_i \omega_i - \sum_{i=1}^k v_i \alpha_i)$  in the tensor product  $\bigotimes_{i=1}^k L(\omega_i)^{\otimes w_i}$  of fundamental representations  $L(\omega_i)$  of  $\mathfrak{gl}(k)_{\text{aff}}$ .*

**7.3. Interpretation via  $\text{SL}(N)$ .** Let  $G = \text{SL}(N)$ . Recall the bijection  $\Psi_{N,k}$  of Section 3.2 between  $\Lambda_k^+$  and the conjugacy classes of homomorphisms  $\Gamma_k \rightarrow \text{SL}(N)$ , i.e. isomorphism classes of  $N$ -dimensional  $\Gamma_k$ -modules of determinant 1. We will write down the elements  $\bar{\mu} \in \Lambda_k^+$  as generalized Young diagrams, i.e. the sequences of integers  $(\mu_1 \geq \mu_2 \geq \dots \geq \mu_N)$  such that  $\mu_1 - \mu_N \leq k$ , and  $\mu_1 + \dots + \mu_N = 0$ .

**Lemma 7.4.** *If  $\Psi_{N,k}(\bar{\mu}) = (w_1, \dots, w_k)$ , then  $w_i$  equals the number of entries among  $\{\mu_1, \dots, \mu_N\}$  congruent to  $i$  modulo  $k$ , for any  $i \in \mathbb{Z}/k\mathbb{Z}$ .  $\square$*

If we view  $(w_1, \dots, w_k)$  as a dominant weight  $w_1 \omega_1 + \dots + w_{k-1} \omega_{k-1}$  of  $\text{PSL}(k)$  then there is a unique way to write it as a generalized Young diagram  $\bar{\nu} = (\nu_1 \geq \nu_2 \geq \dots \geq \nu_k)$  such that  $\nu_i - \nu_{i+1} = w_i$  for any  $1 \leq i \leq k-1$ , and  $\nu_1 - \nu_k \leq N$ , and  $\nu_1 + \dots + \nu_k = 0$ . If  $(w_1, \dots, w_k) = \Psi_{N,k}(\bar{\mu})$ , we will write  $\bar{\nu} = {}^t \bar{\mu}$  (transposed generalized Young diagram). For the corresponding level  $N$  weight of  $\widehat{\mathfrak{gl}}(k)$  we have  $\sum_{i=1}^k w_i \omega_i = \sum_{j=1}^N \omega_{\mu_j}$ . Here  $\omega_{\mu_i}$  is understood as  $\omega_{\mu_i \pmod k}$ .

Similarly, let us consider the level  $N$  dominant  $\mathfrak{gl}(k)_{\text{aff}}$ -weight  $\sum_{i=1}^k w_i \omega_i - \sum_{i=1}^k v_i \alpha_i$ . Its projection to  $\widehat{\Lambda}$  can be written as  $\sum_{i=1}^k w'_i \omega_i$ ; note that  $\sum_{i=1}^k w'_i = \sum_{i=1}^k w_i = N$ . We now consider the corresponding generalized Young diagram  $\bar{\lambda}$  such that  $\Psi_{N,k}(\bar{\lambda}) = (w'_1, \dots, w'_k)$  and we view  $\bar{\lambda}$  as a dominant weight of  $\widehat{\mathfrak{sl}(N)}$  at level  $k$ . Let us now introduce  $\mu, \lambda \in \Lambda_{\text{aff},k}^+$  by setting

$$\mu = (k, \bar{\mu}, -\frac{a + \frac{(\bar{\mu}, \bar{\mu})}{2} - \frac{(\bar{\lambda}, \bar{\lambda})}{2}}{k}), \quad \lambda = (k, \bar{\lambda}, 0).$$

Then we have the following

**Lemma 7.5.** *There exist natural isomorphisms  $\mathfrak{M}_0^{\text{reg}}(V, W) \simeq \text{Bun}_{G,\mu}^\lambda$  and  $\mathfrak{M}_0(V, W) \simeq \mathcal{U}_{G,\mu}^\lambda(\mathbb{A}^2/\Gamma_k)$  where  $\lambda$  and  $\mu$  are constructed as above.*

*Proof.* The isomorphism  $\mathfrak{M}_0^{\text{reg}}(V, W) \simeq \text{Bun}_{G,\mu}^\lambda$  is explained in [24]. The second isomorphism follows from the first and from Theorem 5.12 of [4].  $\square$

We now claim that Theorem 7.2 and Lemma 7.5 imply also that  $\dim V_\mu^\lambda = \dim L(\lambda)_\mu$ . In other words we can't prove the full statement of Conjecture 4.14(3) which is essentially equivalent to the equality  $\text{IC}_\mu^\lambda(q) = {}^e C_\mu^\lambda(q)$ , but we can show that  $\text{IC}_\mu^\lambda(1) = {}^e C_\mu^\lambda(1)$ . From this, Conjecture 4.14(1) follows for free since the nonemptiness of  $\text{Bun}_{G,\mu}^\lambda$  is equivalent to the nonvanishing of  $\text{IC}_\mu^\lambda(1)$ , and hence the geometric and representation-theoretic strings coincide. Moreover, the argument below shows as well that the total dimension of  $\text{IC}(\mathcal{U}_{\text{SL}(N),\mu}^\lambda(\mathbb{A}^2/\Gamma_k))$ -stalk at the generic point of  $\mathcal{U}_{\text{SL}(N),\mu}^\nu(\mathbb{A}^2/\Gamma_k)$  equals  $\dim L(\lambda)_\nu$ , which implies Conjecture 4.14(2).

In order to verify  $\dim V_\mu^\lambda = \dim L(\lambda)_\mu$  it is enough to establish the following:

**Proposition 7.6.**

$$\text{Hom}_{\mathfrak{gl}(k)_{\text{aff}}}(L(\sum_{i=1}^k w_i \omega_i - \sum_{i=1}^k v_i \alpha_i), \bigotimes_{i=1}^k L(\omega_i)^{\otimes w_i}) \simeq L(\lambda)_\mu \quad (7.1)$$

**Warning.** The notation in the formulation of Proposition 7.6 is a little bit misleading: the reader should remember that the integrable modules appearing in the left hand side of (7.1) are modules over  $\mathfrak{gl}(k)_{\text{aff}}$ , and the module  $L(\lambda)$ , which appears in the right hand side, is a module over  $\mathfrak{sl}(N)_{\text{aff}}$  of level  $k$ .

*Proof.* In effect, this is an instance of I. Frenkel's level-rank duality (see [12]). We will follow the notations of Appendix A.1 of [25] (for a detailed exposition see [14]). Inside  $\mathfrak{gl}(kN)_{\text{aff}}$  we consider the subalgebra  $\mathfrak{gl}(k)(N)$  (see [12], §1.2, page 83) which is  $\widehat{\mathfrak{gl}(k)} + \widehat{\mathfrak{sl}(N)}$  with the common central extension (and no loop rotation). The restriction of the basic  $\mathfrak{gl}(kN)_{\text{aff}}$ -module (that is, degree 0 part of the semiinfinite wedge module) to this subalgebra has a simple spectrum ([12], Proposition 2.2). Moreover, it is shown in [12] that:

a) As a  $\widehat{\mathfrak{gl}}(k)$ -module, this basic module is a sum of tensor products of  $N$ -tuples of fundamental  $\widehat{\mathfrak{gl}}(k)$ -modules (*loc. cit.*). This decomposition into direct sum of tensor products corresponds to the decomposition into weight spaces of the Cartan subalgebra of  $\mathfrak{sl}(N) \subset \widehat{\mathfrak{sl}}(N)$  in such a way that the module  $\bigotimes_{i=1}^k L(\omega_i)^{\otimes w_i}$  over  $\widehat{\mathfrak{gl}}(k)$  corresponds to the weight  $\bar{\mu}$  of  $\mathfrak{sl}(N)$  where  $\bar{\mu}$  and  $w$  are related as above.

b) As a  $\widehat{\mathfrak{gl}}(k) + \widehat{\mathfrak{sl}}(N)$  module the basic representation of  $\mathfrak{gl}(kN)_{\text{aff}}$  is isomorphic to the direct sum of modules of the form  $L(\sum_{i=1}^k w_i \omega_i - \sum_{i=1}^k v_i \alpha_i) \otimes L(\lambda)$  where  $v$  and  $\lambda$  are related as above (here the first factor is a  $\widehat{\mathfrak{gl}}(k)$ -module and the second factor is a  $\widehat{\mathfrak{sl}}(N)$ -module).

This immediately proves Proposition 7.6 with the loop rotation action disregarded. In other words, we get an isomorphism

$$\text{Hom}_{\widehat{\mathfrak{gl}}(k)}(L(\sum_{i=1}^k w_i \omega_i - \sum_{i=1}^k v_i \alpha_i), \bigotimes_{i=1}^k L(\omega_i)^{\otimes w_i}) \simeq L(\lambda)_{\bar{\mu}} \quad (7.2)$$

We need to check that the decomposition of both sides of (7.2) with respect to the loop rotation gives rise to the isomorphism (7.1). This is nothing but the formulae (A.6, A.7) of [25]. In effect, in notations of *loc. cit.*, we have to check that

$$\langle d^X, \lambda - \mu \rangle = \frac{a + \frac{\langle \bar{\mu}, \bar{\mu} \rangle}{2} - \frac{\langle \bar{\lambda}, \bar{\lambda} \rangle}{2}}{k} \quad (7.3)$$

According to (A.7) of *loc. cit.*, we have  $\langle d^X, \lambda - \mu \rangle = v_k + \langle d, M(\bar{\lambda}) \rangle - \langle d, M(\bar{\mu}) \rangle$ . Here  $\langle d, M(\bar{\mu}) \rangle$  is the energy level in the semiinfinite wedge  $\mathfrak{gl}(kN)_{\text{aff}}$ -module of the highest vector of the tensor product  $\bigotimes_{p=1}^N L(\omega_{\mu_i})$  of the fundamental  $\mathfrak{gl}(k)_{\text{aff}}$ -modules.

**Lemma 7.7.**  $\langle d, M(\bar{\mu}) \rangle$  (resp.  $\langle d, M(\bar{\lambda}) \rangle$ ) is the sum of negative entries of the corresponding generalized Young diagram  $\bar{\mu}$  (resp.  $\bar{\lambda}$ ).  $\square$

Adding a multiple of the imaginary root  $\delta = (1, \dots, 1)$  to  $v = (v_1, \dots, v_k)$ , we reduce the proof of (7.3) to the case  $v_k = 0$ . In this case  $a = \langle \check{\rho}_{\text{SL}(k)}, {}^t \bar{\mu} - {}^t \bar{\lambda} \rangle$ . Here  $\check{\rho}_{\text{SL}(k)} = \frac{1}{2}(k-1, k-3, \dots, 3-k, 1-k)$ . Thus it remains to check  $\langle \check{\rho}_{\text{SL}(k)}, {}^t \bar{\lambda} \rangle = -\frac{\langle \bar{\lambda}, \bar{\lambda} \rangle}{2} - k \langle d, M(\bar{\lambda}) \rangle$ . Recall that  ${}^t \bar{\lambda}$  is the sum of fundamental weights  $\omega_b$  of  $\text{SL}(k)$  (and  $\omega_b$  is understood as  $\omega_{b \pmod k}$ ). If  ${}^t \bar{\lambda}_b = \omega_b$ , and  $b > 0$ , then  $\langle \check{\rho}_{\text{SL}(k)}, {}^t \bar{\lambda}_b \rangle = \frac{1}{2}(bk - b^2)$ , and if  $b < 0$ , then  $\langle \check{\rho}_{\text{SL}(k)}, {}^t \bar{\lambda}_b \rangle = \frac{1}{2}(-bk - b^2)$ . Summing up these equalities over all  $b$ , we obtain  $\langle \check{\rho}_{\text{SL}(k)}, {}^t \bar{\lambda} \rangle = -\frac{1}{2} \sum_b b^2 + \frac{1}{2} \sum_{b>0} bk - \frac{1}{2} \sum_{b<0} bk = -\frac{\langle \bar{\lambda}, \bar{\lambda} \rangle}{2} - k \sum_{b<0} b = -\frac{\langle \bar{\lambda}, \bar{\lambda} \rangle}{2} - k \langle d, M(\bar{\lambda}) \rangle$  (the last equality holds by Lemma 7.7). This completes the proof of Proposition 7.6.  $\square$

**7.8. Dimensions.** In this section we prove Theorem 4.8. We assume that  $\mathbf{S} = \mathbb{P}^2$ , and pick up a  $G$ -bundle  $\mathcal{F} \in \text{Bun}_{G, \mu}^\lambda = \text{Bun}_{G, \bar{\mu}}^{\bar{\lambda}, a}$  in notations of Definition 4.7. We have  $\dim \text{Bun}_{G, \mu}^\lambda = \dim T_{\mathcal{F}} \text{Bun}_{G, \mu}^\lambda$ . Furthermore, we have  $T_{\mathcal{F}} \text{Bun}_{G, \mu}^\lambda = H^1(\mathbb{P}^2, \text{ad}_{\mathcal{F}}(-1))^{\Gamma_k}$  where  $\text{ad}_{\mathcal{F}}$  stands for the  $\Gamma_k$ -equivariant vector bundle associated to  $\mathcal{F}$  and the adjoint representation of  $G$ . Now  $\text{ad}_{\mathcal{F}}$  can be described in the ADHM terms of the cyclic  $A_{k-1}$ -quiver, and it has



certain dimension vectors  $v = (v_1, \dots, v_k)$  and  $w = (w_1, \dots, w_k)$  attached to it. We have  $\dim H^1(\mathbb{P}^2, \text{ad}_{\mathcal{F}}(-1))^{\Gamma_k} = v_k$ . So we must prove  $2|\lambda - \mu| = v_k$ .

Recall that  $2|\lambda - \mu| = \langle 2\check{\rho}_G, \bar{\lambda} - \bar{\mu} \rangle + 2\check{h}(l - m) = \langle 2\check{\rho}_G, \bar{\lambda} - \bar{\mu} \rangle + \frac{2\check{h}}{k}(a + \frac{\langle \bar{\mu}, \bar{\mu} \rangle}{2} - \frac{\langle \bar{\lambda}, \bar{\lambda} \rangle}{2})$  where  $\check{h}$  is the dual Coxeter number of  $G$ . Note that  $\text{ad}_{\mathcal{F}} \in \text{Bun}_{\text{SL}(\mathfrak{g}), \text{ad}_{\mathcal{F}} \bar{\mu}}^{\text{ad}_{\mathcal{F}} \bar{\lambda}, 2\check{h}a}$  where  $\text{ad}_{\mathcal{F}} \bar{\lambda}$  and  $\text{ad}_{\mathcal{F}} \bar{\mu}$  are the coweights (in fact, coroots) of  $\text{SL}(\mathfrak{g})$ . We have  $(\text{ad}_{\mathcal{F}} \bar{\lambda}, \text{ad}_{\mathcal{F}} \bar{\lambda}) = 2\check{h}(\bar{\lambda}, \bar{\lambda})$ , and  $(\text{ad}_{\mathcal{F}} \bar{\mu}, \text{ad}_{\mathcal{F}} \bar{\mu}) = 2\check{h}(\bar{\mu}, \bar{\mu})$  since the Dynkin index of the adjoint representation of  $G$  is  $2\check{h}$ . Furthermore, if we write down  $\text{ad}_{\mathcal{F}} \bar{\lambda}$  (resp.  $\text{ad}_{\mathcal{F}} \bar{\mu}$ ) as a generalized Young diagram (with  $\dim \mathfrak{g}$  entries summing up to zero) then  $\langle 2\check{\rho}_G, \bar{\lambda} \rangle$  (resp.  $\langle 2\check{\rho}_G, \bar{\mu} \rangle$ ) equals the sum of positive entries of the corresponding Young diagram, which will be denoted  $-\langle d, M(\text{ad}_{\mathcal{F}} \bar{\lambda}) \rangle$  (resp.  $-\langle d, M(\text{ad}_{\mathcal{F}} \bar{\mu}) \rangle$ ) in accordance with Lemma 7.7.

All in all, we must prove

$$v_k = \langle d, M(\text{ad}_{\mathcal{F}} \bar{\mu}) \rangle - \langle d, M(\text{ad}_{\mathcal{F}} \bar{\lambda}) \rangle + \frac{1}{k} \left( c_2(\text{ad}_{\mathcal{F}}) + \frac{(\text{ad}_{\mathcal{F}} \bar{\mu}, \text{ad}_{\mathcal{F}} \bar{\mu})}{2} - \frac{(\text{ad}_{\mathcal{F}} \bar{\lambda}, \text{ad}_{\mathcal{F}} \bar{\lambda})}{2} \right).$$

However, this is exactly what was established in the proof of Proposition 7.6, compare (7.3) and the next line after it. This completes the proof of Theorem 4.8.

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